Order Number 9307617

Extension of Self-Dual Yang-Mills equations across the 8^{th} dimension

Weinstein, Eric Ross, Ph.D.

Harvard University, 1992

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

HARVARD UNIVERSITY

THE GRADUATE SCHOOL OF ARTS AND SCIENCES

THESIS ACCEPTANCE CERTIFICATE

(To be placed in Original Copy)

The undersigned, appointed by the

Division

Department of Mathematics

Committee

have examined a thesis entitled

Extension of Self-Dual Yang-Mills Equations
Across the Eighth Dimension

presented by Eric R. Weinstein

candidate for the degree of Doctor of Philosophy and hereby certify that it is worthy of acceptance.

Signature Raoul Bott

Typed name Raoul Bott

Signature Dror Bar-Natan

Typed name Dror Bar-Natan

Signature ....

Typed name ....

Date September 24, 1992

Extension of Self-Dual Yang-Mills Equations Across The 8^{th} Dimension

A thesis presented

by

Eric Ross Weinstein

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of

Mathematics

Harvard University
Cambridge, Massachusetts

September 1992

Abstract

We introduce a class of elliptic generalized Einstein equations adapting the Self-Dual and Anti-Self-Dual Yang-Mills equations to oriented Riemannian 8-manifolds (X^8, g_{ij}) with the virtual dimension of the Moduli space of solutions given by $Dim(\mathcal{M}(X^8)) = \frac{9\chi(X^8) \mp \sigma(X^8)}{2} \pm 1024\hat{A}(X^8)$. We construct on S^8 a 9-dimensional moduli space $\mathcal{M}(S^8) \cong B^9$ of soliton-like solutions given as the translates of the Levi-Civita connection by arbitrary conformal transformations. Existence is shown on any Einstein manifold. Proposed extension to all even dimensions is sketched.

Contents

1 Introduction **3**

2 Basic Material and Notation **7**
- 2.1 Objects from Representation theory 7
- 2.2 Geometric Objects 12
- 2.3 Fundamental Solutions of the Self-Dual Yang Mills Equations 15

3 Further Constructs **17**
- 3.1 Representations 17
- 3.2 Geometric Constructions 22

4 Equations **27**

5 Solutions **33**
- 5.1 Conformal Solutions on S^8 33
- 5.2 Abstract Solution to manifestly elliptic equations on S^8 36

6 Further Remarks **38**
- 6.1 Amplification of Facts From Representation Theory 38
- 6.2 Proposed Extension of Self-Dual Equations to Even Dimensions 39
- 6.3 Existence of Einstein Structures 43
- 6.4 Odd dimensional companion theories 45
- 6.5 Other Choices of Bundles Governed by Other Elliptic Operators 46
- 6.6 Phenomena Peculiar to Low Dimensions 47
  - 6.6.1 Relativity + Torsion in Dimension 4 48
  - 6.6.2 Other Peculiarities of Torsion and Curvature summands in Low Dimensions for P_{F_r} 50

6.6.3 $SU(3)$-Structures in Dimension 6 51
6.6.4 G_2-Structures in Dimension 7 51
6.6.5 $Spin(7)$-Structures in Dimension 8 51
6.7 Other Elliptic Symbol Sequences 53
6.8 Spin(10)-Phenomenology 58

A Determinants **60**
A.1 Mathematica Code Generating the Symbol Matrix 60
A.2 Example of an Input File Program 67
A.3 Determinants For Various Sets of Constants 67

Chapter 1

Introduction

In the mid 1970's, particle theorists brought to the attention of the mathematics community a new way of looking at the Hopf fibration

$$\begin{array}{ccc} S^7 & \hookleftarrow & S^3 \\ \pi \downarrow & & \\ S^4 & & \end{array} \tag{1.1}$$

introduced decades earlier by Hopf as generating one of the first non-trivial higher homotopy groups $\Pi_{n+i}(S^n)$. They found that the usual metric on the total space S^7 could be viewed as giving a natural symmetric solution to the (anti)-self-dual Yang-Mills equations for the field strength of an $Sp(1)$-gauge-potential in euclidean 4-space. These equations state that for a connection A on a principal bundle P_G over a Riemannian 4-manifold, the curvature must satisfy

$$F_A = \pm * F_A. \tag{1.2}$$

In fact, the translates of this symmetric connection by conformal dilations yield a 5-parameter moduli space of concentrated soliton solutions (called instantons).

Over the last 15+ years, a general theory built on this original set of solutions has been developed by Atiyah, Donaldson, Hitchen, Singer, Taubes, Uhlenbeck and others to investigate moduli spaces of instantons on various spaces. It is clear from 1.2 that the equations as stated are peculiar to dimension 4 where the star operator takes $\Omega^2(T^*X^8)$ into itself.

The purpose of this report is to show that it is possible to move the structures surrounding the self-dual Yang-Mills equations, often thought peculiar to dimension 4, to other dimensions in a meaningful way such that the theory generalizes in the obvious fashion for the fundamental example of the sphere. We carry this out for the case of dimension 8 and will sketch a program for extension to arbitrary even dimension.

Our idea runs along the following lines. We associate to our oriented Riemannian 8-manifold (X^8, g_{ij}) a particular pair of principal bundles $P^{\pm}$ associated via the structure bundle P_{F_r} of orthonormal frames. These have the property that their adjoint bundles $Ad(P^{\pm})$ are actually twisted spin bundles ($S^{\pm} \otimes S^{\pm}$ to be precise). Then to any connection $A \in \mathcal{A}(P^{\pm})$ we wish to associate a collection of three separate tensor fields: a generalized torsion tensor $\tau(A) \in \Omega^1(T^*X) \otimes Ad(P^{\pm})$, a curvature tensor $F_A \in \Omega^2(T^*X) \otimes Ad(P^{\pm})$ and a stress-energy like tensor $T_A \in S^2(T^*X) \otimes Ad(P^{\pm})$.

We then propose a pair of representations of $Spin(8)$, which we will denote by $R^{\pm}$, together with injections $\Phi = \Phi^0 \oplus \Phi^1$ of representations

$$\Phi^0 : S^{\pm} \otimes R^{\pm} \hookrightarrow \Lambda^1(\mathbb{R}^8) \otimes S^{\pm} \otimes S^{\pm} \tag{1.3}$$

$$\Phi^1 : S^{\mp} \otimes R^{\pm} \hookrightarrow \Lambda^1(\mathbb{R}^8) \otimes \Lambda^1(\mathbb{R}^8) \otimes S^{\pm} \otimes S^{\pm} \tag{1.4}$$

determining bundle maps over the manifold in the natural way. With this established, we put forward the system of equations

$$\Pi_{Im(\Phi^0)}(\tau(A)) = \tau(A) \tag{1.5}$$

$$\Pi_{Im(\Phi^1)}(F_A \oplus T_A) = 0 \tag{1.6}$$

which will be a system of elliptic 1^{st} order non-linear PDE's in $\mathcal{A}(X^8)$ subject to a 0^{th} order (linear) constraint equation combining aspects of the Einstein, Dirac and Yang-Mills equations in a Riemannian setting.

The plan is as follows. In chapter 2 we review standard notions from Representation theory, Geometry, and Gauge theory and use the opportunity to set notation. In chapter 3 we specialize the representation theory to the case at hand and develop the tools necessary to define our equations. After explaining the necessary algebra, we introduce the geometric constructions needed to define the equations. In chapter 4 we define the equations and compute the index of the twisted dirac operator on which our equations build. It is here that we also show how our equations reduce to the vacuum

Einstein equations in the absence of torsion. This gives us an automatic existence result on any manifold equipped with an Einstein metric. In chapter 5 we solve the equations for a specific choice of Φ. In this case we readily show that the moduli space $\mathcal{M}(S^8)$ contains the pull backs of the (extended) Levi-Civita connection under conformal dilation on the sphere as in Dimension 4. The symbol of the our operator can easily be turned into a matrix by choosing bases for the representations used to associate our bundles; as this matrix is 448×448 we allow the computer the privilege of checking that the determinant is indeed non-zero (which it is for a generic choice of Φ). It has been mentioned to the author (by D. Bar-Natan and M. Grossberg) that it is perhaps misleading not to explain the extension of our techniques to other even dimensions and G-structures. This is done informally in chapter 6. The Mathematica program referred to above written by Bar-Natan and the Author (the code by Bar-Natan, algebra by the author) is given in the appendix along with several elliptic choices of Φ (and one that is not for contrast).

Acknowledgements: I would like to thank **A. Abdulali, D. Bar-Natan,** J. Bernstein, **R. Bott**, J. Bryan, E. Calabi, G. Comer, S. Coleman, M. Du Sautoy, S. Elitzur, N. Elkies, P. Freyd, H. Georgi, H. Gluck, **M. Grossberg**, R. Iyer, R. Jackiw, Y. Karshon, **D. Kazhdan**, B. Kostant, R. Kuhn, P. Magyar, G. Matic, R. Melrose, **H. Miller**, T. Mrowka, V. Pati, S. Petrack, J. Primer, D. Roberts, S. Schatz, W. Schmid, **I. Singer**, J. Stasheff, **S. Sternberg**, A. Szenes, E. Tweitmeyer, D. Vogan, M. Zabcic and W. Ziller for numerous discussions.

I would like to remind S. Glashow that he owes me a dollar. See [10].

I would also like to thank the Office of Naval Research, Harvard University and my parents for significant financial support during the course of my studies. Almost all of the actual research in this work was carried out during 1989-1990 or 1991-1992 during which time the author received financial assistance from his parents, and later The Racah institute of Physics and the Einstein institute of Mathematics in The Hebrew University of Jerusalem.

I would like to thank Pia Malaney for support beyond what one person can ask from another and for what cannot be said in print. I would like to thank Dror Bar-Natan for being a real Mensch as well as a good friend. I would further like to express my deep gratitude to my parents, Karen and Les Weinstein, my closest friends (Adil and Mike as above), Geoff Alperin, Bret Weinstein, Sophie Rubin, Caroline Appfel, Judy and Herman Lippman, Steve, Frederique, Arnold, and Phylis Marglin, Dan Abramovich, Toscun Aricanli, all those who I will kick myself later for forgetting, Nadja Kutz, Murat Ucer, Kiev(Alla, Igor, Rachel, Olga), Moscow (Tanya, Alec, Eugene), Voroshilovgrad/Lugansk/Ariel/Rehovot (Misha and Natasha), Marcus Du Sautouy, Shani Wolfensberger, Barbara Brody, Gwen Freyd, and the Extended Weinstein, Karvarsarsky, Rubin, Kor, Lipman, Marglin, and Abdulali families of Israel, Russia, Ukraine, India, and at home for providing me with real family wherever I've found myself.

The Ideas contained in this work are all dedicated to my one great teacher: Chemist, Philosopher, Patent Holder, Junior College Graduate, Punster, Mandolin player, Naturalist, Socialist, Humanist, and luckily for me, my Grandfather, Harry Rubin (going strong at 79). Thanks Harry for your continued unjustifiable confidence in me over the years. I'm sorry it took so long but I ran into some of what you warned me about along the way. I only wish you could understand these Ideas enough to see how much of your spirit lives within these pages. I think you'd be pleased.

Chapter 2

Basic Material and Notation

Our approach rests on the correspondence between between tensor bundles of a Riemannian manifold, and the representation theory of its structure group (in this case $Spin(8)$ or $SO(8)$); we discuss this below in the next two sections (see e.g. [13]).

2.1 Objects from Representation theory

We begin with a useful definition defining the natural product operation on the semigroup of irreducible highest weight representations of a reductive group G.

Definition 1 *Given two representations W_μ, W_ν of highest weights μ and ν we define the Cartan product by*

$$C(W_\mu, W_\nu) := W_{\mu+\nu} \tag{2.1}$$

and will define the Cartan powers of W_ν by

$$C^p(W_\nu) := W_{p\nu} \tag{2.2}$$

Note: $C(W_\mu, W_\nu)$ is always a summand of $W_\mu \otimes W_\nu$ and always appears with multiplicity one.

We will find it convenient to keep track of $Spin(8)$-modules/associated vector bundles, by their highest weights. It is well known that via the Borel-Weil-Bott construction, any irreducible representation of a simply connected compact simple Lie group G can be constructed directly from its heighest weight. This establishes a constructive isomorphism from the semi-group of lattice points in the weight lattice with positive coordinates to the semi-group of representations (under the above product operation).

Let $\{e_i\}_{i=1}^4$ be the usual basis for $\mathbb{R}^4$ thought of as the dual Lie-algebra to a maximal torus for $Spin(8)$. A positive root system of 12 roots is given by $\{e_i \pm e_j\}$ $1 \leq i < j \leq 4$. The weight lattice is then generated by the highest weights of the fundamental representations. It will be more convenient for us in what follows to specify elements of the weight lattice as 4-tuples (x_1, x_2, x_3, x_4) corresponding to the natural weight basis.

The fundamental representations of $Spin(8)$ are the defining representation of $SO(8)$ (denoted in what follows by any of $T \cong T^* \cong \Lambda^1(T^*) \cong \Lambda^1 \cong \mathbb{R}^8$), the adjoint representation $\Lambda^2(T^*)$, and each of the two chiral semi-spin representations S^+, S^-.

They are given (via their highest weights in our notation (x_1, x_2, x_3, x_4)) as

$$(1,0,0,0) = e_1 = \lambda(\mathbb{R}^8) \tag{2.3}$$

$$(0,1,0,0) = e_1 + e_2 = \lambda(\Lambda^2(\mathbb{R}^8)) \tag{2.4}$$

$$(0,0,1,0) = (e_1 + e_2 + e_3 + e_4)/2 = \lambda(S^+(\mathbb{R}^8)) \tag{2.5}$$

$$(0,0,0,1) = (e_1 + e_2 + e_3 - e_4)/2 = \lambda(S^-(\mathbb{R}^8)) \tag{2.6}$$

So a representation (x_1, x_2, x_3, x_4) is given by $y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4$ where

$$\begin{aligned} y_1 &= x_1 + x_2 + (x_3 + x_4)/2 \\ y_2 &= x_2 + (x_3 + x_4)/2 \\ y_3 &= (x_3 + x_4)/2 \\ y_4 &= (x_3 - x_4)/2 \end{aligned} \tag{2.7}$$

The Weyl Dimension Formula states that for a compact semi-simple Lie group G, the dimension of an irreducible G-module V_λ of highest weight λ is given by a particular polynomial in the x_i called the dimension polynomial.

Specifically:

Theorem 2 *(Weyl) Let G be a compact simply connected semi-simple Lie group with a system of positive roots Δ^+ and let ϱ denote half the sum of the positive roots. Then if V_λ is an irreducible G-module of highest weight λ the dimension is given as*

$$Dim(V_\lambda) = \frac{\prod_{\alpha\in\Delta^+} < \lambda + \varrho, \alpha >}{\prod_{\alpha\in\Delta^+} < \varrho, \alpha >} \tag{2.8}$$

(See [13]). In our case this, this gives the explicit formula for $Spin(8)$:

Corollary 3 *For the irreducible $Spin(8)$ representation V_μ of highest weight $\mu = ((x_1, x_2, x_3, x_4))$ in the canonical basis of the weight lattice we have:*

$$Dim((x_1, x_2, x_3, x_4)) = (y_1 + y_2 + 5)(y_1 - y_2 + 1)(y_1 + y_3 + 4) \tag{2.9}$$

$$(y_1 - y_3 + 2)(y_1 + y_4 + 3)(y_1 - y_4 + 3)(y_2 + y_3 + 3)(y_2 - y_3 + 1)$$

$$(y_2 + y_4 + 2)(y_2 - y_4 + 2)(y_3 + y_4 + 1)(y_3 - y_4 + 1)/4320$$

with the y_i defined as defined in 2.7.

We recall the weights of other important non-fundamental representations which appear frequently.

Fact 4 *In terms of the generators of the weight lattice for $Spin(8)$, further important representations appear via their highest weights as*

$$(0, 0, 1, 1) = \lambda(\Lambda^3(T^*\mathbb{R}^8)) \tag{2.10}$$

$$(0, 0, 2, 0) = \lambda(\Lambda^4_+(T^*\mathbb{R}^8)) \tag{2.11}$$

$$(0, 0, 0, 2) = \lambda(\Lambda^4_-(T^*\mathbb{R}^8)) \tag{2.12}$$

$$(2, 0, 0, 0) = \lambda(S^2_0(T^*\mathbb{R}^8)) \tag{2.13}$$

$$(0, 2, 0, 0) = \lambda(Weyl(T^*\mathbb{R}^8)) \tag{2.14}$$

where $Weyl(T^*\mathbb{R}^8)$ is the representation corresponding to the Weyl curvature tensor living in $\Omega^2(T^*X) \otimes \Omega^2(T^*X)$ as the curvature summand derived from the Cartan square $C^2(\Lambda^2(\mathbb{R}^n))$.

We now wish to describe a few of the basic properties of the (semi)-Spin representation(s) of $Spin(2n)$. Thorough discussions can be found in [2, 12, 15].

First of all, recall that the usual $\mathbb{Z}_2$-graded Clifford algebra $cl(\mathbb{R}^{2n}) = cl_0(\mathbb{R}^{2n}) \oplus cl_1(\mathbb{R}^{2n})$ of $\mathbb{R}^{2n}$ with the standard metric is a natural $SO(n)$ module isomorphic (at the level of representation theory) to the exterior algebra $\Lambda^*(\mathbb{R}^{2n})$. Consider the subspace of the even sub-algebra $U \subset cl_0$ defined by

$$U = \bigoplus_{j=0} \Lambda^{2+4j}(\mathbb{R}^{2n}). \tag{2.15}$$

Under the commutator operation in the Clifford algebra, this subspace becomes a Lie algebra. Specifically, if we write $n = 4k + l$ (where $k, l \in \mathbb{Z}$ are non-negative with $l \leq 3$), then $(U, [\cdot, \cdot])$ is isomorphic to $\mathfrak{sp}(2^{4k}) \oplus \mathfrak{sp}(2^{4k})$ for $l = 2$, $\mathfrak{so}(2^{4k-1}) \oplus \mathfrak{so}(2^{4k-1})$ for $l = 0$ and $\mathfrak{u}(2^{4k+l-1})$ for $l = 1, 3$.

The natural Spin representation of $\mathfrak{so}(2n)$ is then just the inclusion homomorphism from $\iota : \Lambda^2 \longrightarrow U$ acting on the fundamental modules $S(\mathbb{R}^{2(4k+l)}) = S^+ \oplus S^-$ where $S = \mathbb{H}_+^{4k} \oplus \mathbb{H}_-^{4k}$, $\mathbb{R}_+^{4k-1} \oplus \mathbb{R}_-^{4k-1}$, or $\mathbb{C}_+^{4k+l-1} \oplus \bar{\mathbb{C}}_-^{4k+l-1}$ respectively. This cannot be extended to the group $SO(n)$ but only to its simply connected double cover $Spin(n)$.

Upon complexification of the Spin modules, we recover the full Clifford algebra as $cl(\mathbb{R}^{2n}) \cong \mathfrak{u}(S_{\mathbb{C}})$. In the case when $n = 4k$ we have the following isomorphisms

$$\mathfrak{u}(S_{\mathbb{C}}^{\pm}) \cong S^{\pm} \otimes S^{\pm} \cong \bigoplus_{0 \leq i \leq \frac{n-1}{2}} \Lambda^{2i}(T) \oplus \Lambda_{\pm}^{n} \tag{2.16}$$

Note: These isomorphisms are abstract equivalences and not isomorphisms with sub-representations of $cl(\mathbb{R}^n)$.

$$S^+ \otimes S^- \cong \bigoplus_{1 \leq i \leq l} \Lambda^{2i-1}(T) \tag{2.17}$$

giving for $k = 1$ the decompositions (cf. [12, 15]):

Fact 5

$$S^{\pm}(\mathbb{R}^8) \otimes S^{\pm}(\mathbb{R}^8) \cong \Lambda^0(\mathbb{R}^8) \oplus \Lambda^2(\mathbb{R}^8) \oplus \Lambda_{\pm}^4(\mathbb{R}^8) \tag{2.18}$$

Fact 6

$$S^{\pm}(\mathbb{R}^8) \otimes S^{\mp}(\mathbb{R}^8) \cong \Lambda^1(\mathbb{R}^8) \oplus \Lambda^3(\mathbb{R}^8) \tag{2.19}$$

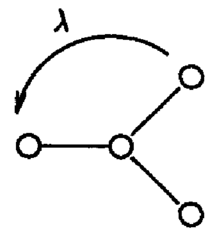

Figure 2.1: D_4 Dynkin Diagram

We note that the inclusion of $\Lambda^1(\mathbb{R}^n) \hookrightarrow cl_1(\mathbb{R}^n)$ yields a map

$$c : T^* \otimes S \longrightarrow S \tag{2.20}$$

for which we have $ker(c(\nu, \cdot)) \neq 0$ iff $\nu = 0$. This is the reason for the ellipticity of the Dirac operator which we discuss in the following section.

Lastly we would like to point out

Fact 7 *$S^\mp$ appears with multiplicity 1 in $T^* \otimes S^\pm$.*

We will show that this follows from an elementary dimension counting argument in the next chapter for the case of Spin(8) but it is equally valid in all even dimensions.

In the case $k = 1$, $l = 0$, we have an added benefit that there exists an outer automorphism

$$\lambda : Spin(8) \longrightarrow Spin(8) \tag{2.21}$$

of order 3, which cyclically permutes the three eight dimensional representations T, S^+, S^-. This is visualized in terms of the D_4 Dynkin diagram in figure 2.1 as counter clockwise rotation by $\frac{2\pi}{3}$. This automorphism can be exploited to avoid redundant calculations and to rotate unfamiliar representations into familiar ones. As an example, λ tells us that

$$T \otimes S^\pm \cong S^\mp \oplus \Lambda^3(S^\mp). \tag{2.22}$$

Fact 8 *Let $\lambda : Spin(8) \longrightarrow Spin(8)$ be an outer automorphism such that λ induces a counter-clockwise rotation of the D_4 Dynkin diagram by $\frac{2\pi}{3}$. Then* *there exists a faithful homomorphism $\iota : G_2 \hookrightarrow Spin(8)$ of the 14-dimensional Lie group G_2 into the 28-dimensional $Spin(8)$ such that fixed point set of λ is precisely $\iota(G_2)$; further, given any 3 irreducible highest weight representations $\rho_i : Spin(8) \longrightarrow Aut(V_{\mu_i})$ $i = 0,1,2$ with $\lambda(\mu_1) = \lambda^2(\mu_2) = \mu_0$, then there exist maps $\hat{\lambda}^i : V_{\mu_i} \longrightarrow V_{\mu_0}$ such that the diagram*

$$\begin{array}{ccc} Spin(8) \times V_{\mu_i} & \xrightarrow{\lambda^i \times \hat{\lambda}^i} & Spin(8) \times V_{\mu_0} \\ \rho_i \downarrow & & \rho_0 \downarrow \\ V_{\mu_i} & \xrightarrow{\hat{\lambda}^i} & V_{\mu_0} \end{array} \tag{2.23}$$

commutes (equivalently, for any pair of elements $g \in Spin(8)$ and $v_i \in V_{\mu_i}$, $\rho_0(\lambda^i(\rho_i(g)))\hat{\lambda}(v_i) = \rho_0(\lambda^i(g) \times \hat{\lambda}(v_i))$.

(For details on triality and the octonion algebra, see [7, 11, 12, 15, 17]).

2.2 Geometric Objects

We recall the definitions of the basic objects of the theory of connections in order to fix notation. For our purposes in what follows G will be a compact reductive Lie group.

A principal G-bundle P_G over a manifold X is a manifold admitting a free right G-action such that the quotient of this action is a locally trivial fibration

$$\begin{array}{ccc} P_G & \hookleftarrow & G \\ \pi \downarrow & & \\ X. & & \end{array} \tag{2.24}$$

If $\rho : G \longrightarrow Aut(V)$ is a representation of G on a linear space V then we will denote by $E_\rho \cong P_G \times_\rho V$ the associated vector bundle

$$\begin{array}{ccc} P_G \times_\rho V & \hookleftarrow & V \\ \pi \downarrow & & \\ X & & \end{array} \tag{2.25}$$

obtained from quotienting the Cartesian product $P_G \times V$ by the natural action of G on both factors. We will refer specifically to the vector bundle $P_G \times_{ad} \mathfrak{g}$ associated via the adjoint representation as $Ad(P_G)$. By a connection A on P_G we mean a C^∞ G-invariant distribution of horizontal subspaces of TP_G.

We will denote the space of such connections by $\mathcal{A}(P_G)$ and note that it is naturally an affine space modeled on $\Omega^1(T^*X) \otimes Ad(P_G)$. If $\eta : G \hookrightarrow H$ is a homomorphism of compact reductive groups, then we can form an associated principal H-bundle $P_H \cong P_G \times_\eta H$ in the obvious way. There is then a canonical isomorphism

$$\mathcal{A}(P_H) \cong \mathcal{A}(P_G) \times (\Omega^1(T^*X) \otimes E_\nu) \tag{2.26}$$

where ν is the defined by the action of G on $\mathfrak{g}^\perp \subset \mathfrak{h}$. A nice description of this is spelled out in [18].

Now given a Riemannian manifold (X^n, g_{ij}), the bundle of orthonormal frames P_{Fr} of $TX \cong_{g_{ij}} T^*X$ admits an obvious action by $SO(n)$ giving it the structure of a right principal $SO(n)$-bundle

$$\begin{array}{lcl} P_{Fr} & \hookleftarrow & SO(n) \\ \pi \downarrow & & \\ X^n. & & \end{array} \tag{2.27}$$

In this case all associated vector bundles E to P_{Fr} appear (non-uniquely) as sub-bundles of the bundle of the full tensor algebra with $V \cong \bigoplus_{i=0}^\infty \bigotimes_i T^*X$. In particular $Ad(P_{Fr}) \cong \Lambda^2(T^*X)$. We will denote the bundle of all linear frames of T^*X by P_{fr}.

Given a vector bundle E_ρ associated to P_G via a representation ρ of G, any connection A on P_G canonically determines a (1^{st}-order) linear operator ∇^A called the covariant derivative associated to A of the form $\nabla^A : \Gamma^\infty(E_\rho) \longrightarrow \Omega^1 \otimes (E_\rho)$. In the special case where $P_G = P_{Fr}$ and $E_\rho = TX$, then we can define a map

$$\tau : \mathcal{A}(P_{Fr}) \longrightarrow \Omega^1 \otimes \Omega^2 \tag{2.28}$$

called the torsion mapping via

$$\tau(A)_x(w, u, v) = < w, (\nabla_U V - \nabla_V U - [U, V])_x > \tag{2.29}$$

where U, V are any vector fields extending the vectors $u, v \in T_xX$. The fundamental theorem of Riemannian geometry states that there is a unique connection $A_0 \in \mathcal{A}(P_{Fr})$ with $\tau(A_0) = 0$. This is the well known Levi-Civita connection which we may also denote by A_{L-C}. An important interpretation of the torsion tensor is that the torsion satisfies

$$\tau(A) = A - A_0 \tag{2.30}$$

where we implicitly make use of the above identification $Ad(P_{Fr}) \cong \Lambda^2(T^*X)$.

We briefly recall the definition of curvature for a connection in the setting of principal bundles. Given two G-invariant vector fields $\tilde{U}_i \in \Gamma^\infty(TP_G)$ $i = \{1,2\}$ which are horizontal with respect to a connection $A \in \mathcal{A}(P_G)$ there exist unique vector fields $U_i \in \Gamma^\infty(TX)$ such that $\pi^*(U_i) =_A \tilde{U}_i$ (cf. e.g. [6] where they are refered to as 'basic' vector fields). In this case the expression $Vert([\tilde{U}_1, \tilde{U}_2])_p$ is G-equivariant and independent of the fields extending $(U_i)_p$, and thus defines a section $F_A \in \Omega^2(T^*X) \otimes Ad(P_G)$ where we have made use of the canonical identification $\Gamma^\infty_G(Vert(P_G), P_G) \cong \Gamma^\infty(Ad(P_G), X)$. From this perspective, the curvature of any connection A can be computed relative to a base connection A_0 by the prescription

$$F_A = F_{A_0} + d_{A_0}\alpha + 1/2[\alpha, \alpha] \tag{2.31}$$

where $\alpha = A - A_0$.

Definition 9 *A metric is called Einstein if its Einstein tensor $G_{\mu\nu}$ satisfies*

$$G_{\mu\nu} \equiv R_{\mu\nu} - 1/2sg_{\mu\nu} = \beta g_{\mu\nu} \tag{2.32}$$

where $s : \mathcal{A}(P_{Fr}) \longrightarrow \mathbb{R}$ is the scalar curvature and $\beta \in \mathbb{R}$.

Note: This corresponds in physics to the metric satisfying the Einstein field equation with (possibly) non-zero cosmological constant β:

$$G_{\mu\nu} + \beta g_{\mu\nu} = T_{\mu\nu} \tag{2.33}$$

for the vacuum $T_{\mu\nu} = 0$ where $T_{\mu\nu}$ is the symmetric two tensor describing the matter-energy distribution of the physical system being modeled.

Recall the definition of the full Dirac operator on an even dimensional manifold with Spin-structure P_{Spin}.

Definition 10 *If P_{Spin} is a two-fold cover of P_{Fr} with $Spin(2n)$ as fiber over (X^{2n}, g_{ij}), then the Dirac operator is defined to be*

$$\partial\!\!\!/ = \Pi_S \circ \nabla : \Gamma^\infty(S) \longrightarrow \Gamma^\infty(S) \tag{2.34}$$

which is the sum of the chiral dirac operator

$$\partial\!\!\!/^+ = \Pi_{S^-} \circ \nabla : \Gamma^\infty(S^+) \longrightarrow \Gamma^\infty(S^-) \tag{2.35}$$

and its adjoint.

Recall that on an even dimensional manifold we can (at least locally) express the DeRahm operator relative to the Levi-Civita connection A and a spin structure as

$$\partial\!\!\!/_A : S^+ \otimes S \longrightarrow S^- \otimes S \tag{2.36}$$

where $S = S^+ \oplus S^-$. This in fact can be seen to be the direct sum of two separate elliptic complexes:

$$\partial\!\!\!/_A : S^+ \otimes S^+ \longrightarrow S^- \otimes S^+ \tag{2.37}$$

$$\partial\!\!\!/_A : S^+ \otimes S^- \longrightarrow S^- \otimes S^-. \tag{2.38}$$

In the case of the signature operator, the K-theoretic difference of these 'half-signature' operators yields the signature of the intersection form.

2.3 Fundamental Solutions of the Self-Dual Yang Mills Equations

For a given principal G-bundle P_G, the gauge group $\mathcal{G}(P_G)$ is defined to be the group of automorphism of P_G leaving the fibers invariant. This can be identified with the space $\Gamma^\infty(P_G \times_{Ad} G)$ which acts naturally on $\mathcal{A}(P_G)$ (see [8, 14] for details). For a given automorphism $g \in \mathcal{G}(P_G)$, the effect on the curvature is given by

$$F_{g^*(A)} = Ad_g(F_A) \tag{2.39}$$

where the action on the right hand side is the action of the automorphism group on the Ad bundle (which has no effect on the Λ^2 factor).

In dimension 4, the star operator $*_{g_{ij}} : \Lambda^i \longrightarrow \Lambda^{4-i}$ has eigenvalues ± 1 when restricted to Λ^2 for any g_{ij} of euclidean signature. The self-dual (or anti-self-dual) Yang-Mills equation for a connection $A \in \mathcal{A}(P_G)$ is

$$* F_A = (-) * F_A \tag{2.40}$$

The objects of interest in 4 dimensional gauge theory are the gauge equivalence classes of connections $[A] \in \mathcal{A}(P_G)/\mathcal{G}(P_G)$ satisfying 2.40 (this is well defined by 2.39).

For many purposes it is more convenient to work upstairs on the affine space $\mathcal{A}$, than on the topologically non-trivial quotient $\mathcal{A}/\mathcal{G}$. Let us assume that A is a connection which solves 2.40. Then, as for any connection, the tangent space to oribit of A under the action of $\mathcal{G}$ at A is $d_A(\Omega^0 \otimes Ad(P_G)) \subset \Omega^1 \otimes Ad(P_G)$. Further, the space of infinitesimal deformations which could possibly yield self-dual connections is given as $Ker(\Pi_- \circ d_A)$ where

$$\Pi_- \circ d_A : \Omega^1 \otimes Ad(P_G) \longrightarrow \Omega^2_- \otimes Ad(P_G). \qquad (2.41)$$

Then the space of infinitesimal possible solutions which are perpendicular to the orbit under $\mathcal{G}$ is given as H^1 of the half-signature complex:

$$0 \longrightarrow \Omega^0 \otimes Ad(P_G) \xrightarrow{d_A} \Omega^1 \otimes Ad(P_G) \xrightarrow{\Pi_- \circ d_A} \Omega^2_- \otimes Ad(P_G) \longrightarrow 0 \qquad (2.42)$$

twisted by $Ad(P_G)$.

The keystone of the theory is the moduli space of solutions to 2.40 over S^4 with $P_G = Sp(2)/Sp(1) \cong S^7$. In general, the group of conformomorphisms of S^n is $SO(n+1,1)$ via its natural action on the space of null geodesics in $\mathbb{R}^{n+1,1}$. As 2.40 are manifestly conformally invariant, $Spin(5,1)$ acts naturally on the fundamental solution which is stabilized by the cover of the isometry group $Sp(2) \subset Spin(5,1)$. The quotient $\frac{Spin(5,1)}{Sp(2)} \cong B^5$ is actually the full moduli space for the $Sp(1)$-bundle represented by S^7. Any element of this quotient has a representative dilation by a factor $\lambda \in \mathbb{R}$ of the form $\sigma_p^{-1} \circ \lambda \cdot (\cdot) \circ \sigma_p$, where σ_p is stereographic projection from a point $p \in S^n$.

We now wish to use the above representation of an arbitrary conformal dilation ϕ to generate a family of connections. Consider the natural bundle of frames $P_{F_r}(S^n)$ with the canonical Levi-Civita connection. Recalling that $Ad(P_{F_r}(X)) \cong \Lambda^2(T^*X)$ we have

$$A^\lambda = \sum_{i<j} \frac{(x_i dx_j - x_j dx_i) \otimes dx_i \wedge dx_j}{\lambda^2 + \| r \|^2} \qquad (2.43)$$

which in dimension 4 splits into two pieces as

$$A^\lambda = Im(\frac{x \cdot d\bar{x}}{\lambda^2 + \| r \|^2}) \oplus Im(\frac{\bar{x} \cdot dx}{\lambda^2 + \| r \|^2}) \qquad (2.44)$$

with the usual quaternionic notation $x = x_0 + ix_1 + jx_2 + kx_3$ and $dx = dx_0 + idx_1 + jdx_2 + kdx_3$ with the bar denoting quaternionic conjugation (cf. [1, 8, 14] for the above).

Chapter 3

Further Constructs

3.1 Representations

We now define an interesting series of representations denoted by Y_i which will be important to us in all of what follows.

Definition 11 *If $\iota_v(\phi)$ and $v \wedge \phi$ denote the interior and exterior products of a form $v \in \Lambda^1(\mathbb{R}^n)$ with a form $\phi \in \Lambda^i(\mathbb{R}^n)$ then we define Y_i as a representation of $SO(n)$ via the exact sequence*

$$0 \longrightarrow Y_i \hookrightarrow \Lambda^1 \otimes \Lambda^i \xrightarrow{\theta(\cdot)} \Lambda^{i-1} \oplus \Lambda^{i+1} \longrightarrow 0 \tag{3.1}$$

where $\theta(v \otimes \phi) = \iota_v(\phi) \oplus v \wedge \phi$.

Proposition 12 *$Y_i(\mathbb{R}^8) \cong C(\Lambda^1(\mathbb{R}^8), \Lambda^i(\mathbb{R}^8))$ and is therefore irreducible so long as $i \neq 4$. In this case $Y_4(\mathbb{R}^8) \cong Y_4^+(\mathbb{R}^8) \oplus Y_4^-(\mathbb{R}^8)$ where each of the summands is separately an irreducible $SO(8)$ module.*

Proof: The map $\theta : \Lambda^1 \otimes \Lambda^i \longrightarrow \Lambda^{i-1} \oplus \Lambda^{i+1}$ is obviously surjective. Then we know that neither Λ^{i-1} nor Λ^{i+1} can possibly be the Cartan product $C(\Lambda^1, \Lambda^i)$ as all exterior representations are either fundamental $SO(n)$ representations or summands of the Cartan product $C(S, S)$. Hence we know that $C(\Lambda^1, \Lambda^i) \subset Y_i$. Now we use Cor 3 to compute the dimensions of the $C(\Lambda^1, \Lambda^i)$ and find that representations

$$Dim(Y_1(\mathbb{R}^8)) = Dim((2, 0, 0, 0)) = 35 \tag{3.2}$$

$$Dim(Y_2(\mathbb{R}^8)) = Dim((1,1,0,0)) = 160 \tag{3.3}$$

$$Dim(Y_3(\mathbb{R}^8)) = Dim((1,0,1,1)) = 350 \tag{3.4}$$

$$Dim(Y_4(\mathbb{R}^8)) = Dim((1,0,2,0)) + Dim((1,0,0,2)) = 224+224 = 448 \tag{3.5}$$

and isomorphism follows for Y_i $i = 1,2,3$. This argument goes through word for word in the case of middle dimension with the added requirement that one keep track of the dimension counting for the two halves separately.

Definition 13 *Let $c : T^* \otimes S^{\pm} \longrightarrow S^{\mp}$ denote the Clifford multiplication map of 2.20. This map is obviously not injective by dimensional reasoning and we may thus define a pair of $Spin(2n)$ modules $R^{\pm}$ via the exact sequence*

$$0 \longrightarrow R^{\pm} \hookrightarrow T^* \otimes S^{\pm} \xrightarrow{c} S^{\mp} \longrightarrow 0. \tag{3.6}$$

Proposition 14 *In dimension 8 we have the isomorphism*

$$R^{\pm} \cong \Lambda^3(S^{\mp}) \tag{3.7}$$

and thus $R^{\pm}$ is irreducible.

Proof: Even though the above statement is equivalent to 2.22, the statement that $R^{\pm}$ is irreducible can be demonstrated by a dimension counting argument which we give below; this second proof has the advantage of working in arbitrary even dimensions.

In dimension 8 the Spin representation S is of dimension 16 with the semi-spin representations $S^{\pm}$ of dimension 8. We know that clifford multiplication induces a (non-zero) map of $Spin$-modules $c : T^* \otimes S^{\pm} \longrightarrow S^{\mp}$ which must necessarily be in the form of an orthogonal projection. Since we know that $S^{\mp}$ is a fundamental representation, it cannot be the Cartan product $C(T^*, S^{\pm})$ and thus we must have an inclusion $C(T^*, S^{\pm}) \hookrightarrow R^{\pm}$. Now given 2.3, 2.5 and 2.6, applying Corollary 3 to the Cartan product yields

$$Dim(C(T^*, S^{\pm})) = 56 \tag{3.8}$$

which is $Dim(T^* \otimes S^{\pm}) - Dim(S^{\mp}) = 64 - 8$ so (making use of Triality, and 2.10) we have

$$T^* \otimes S^{\pm} \cong S^{\mp} \oplus \Lambda^3(S^{\mp}) \tag{3.9}$$

as claimed.

Proposition 15 : $S^+ \otimes \Lambda^3(S^-) \cong \Lambda^1(T) \oplus Y_2(T) \oplus \Lambda^3(T) \oplus Y_4^+(T)$

Proof: We proceed at the level of K-theory using the triality principle (Fact 8) to argue in terms of virtual representations.

$$S^+ \otimes \Lambda^3(S^-) \cong S^+ \otimes (S^+ \otimes T \ominus S^-) \tag{3.10}$$

$$\cong (S^+ \otimes S^+) \otimes T \ominus (T \oplus \Lambda^3(T)) \tag{3.11}$$

$$\cong (\Lambda^0(T) \oplus \Lambda^2(T) \oplus \Lambda^4_+(T)) \otimes T \ominus (T \oplus \Lambda^3(T)) \tag{3.12}$$

$$\cong T \oplus (\Lambda^1(T) \oplus \Lambda^3(T) \oplus Y_2(T)) \oplus (\Lambda^3(T) \oplus Y_4^+(T)) \ominus (T \oplus \Lambda^3(T)) \tag{3.13}$$

$$\cong \Lambda^1(T) \oplus Y_2(T) \oplus \Lambda^3(T) \oplus Y_4^+(T) \tag{3.14}$$

Similarly we have:

Proposition 16 $S^- \otimes \Lambda^3(S^-) \cong Y_1(T) \oplus \Lambda^2(T) \oplus Y_3(T) \oplus \Lambda^4_+(T)$

Proof: We proceed as before.

$$S^- \otimes \Lambda^3(S^-) \cong S^- \otimes (S^+ \otimes T \ominus S^-) \tag{3.15}$$

$$\cong T \otimes (S^- \otimes S^+) \ominus (S^- \otimes S^-) \tag{3.16}$$

$$\cong T \otimes (\Lambda^1(T) \oplus \Lambda^3(T)) \ominus (\Lambda^0(T) \oplus \Lambda^2(T) \oplus \Lambda^4_-) \tag{3.17}$$

$$\cong (\Lambda^0(T) \oplus \Lambda^2(T) \oplus Y_1(T)) \oplus (\Lambda^2(T) \oplus Y_3(T) \oplus \Lambda^4_+(T) \oplus \Lambda^4_-(T)) \tag{3.18}$$

$$\ominus (\Lambda^0(T) \oplus \Lambda^2(T) \oplus \Lambda^4_-)$$

$$\cong Y_1(T) \oplus \Lambda^2(T) \oplus Y_3(T) \oplus \Lambda^4_-(T) \tag{3.19}$$

We now look at tensor products of the form $\Lambda^2(\mathbb{R}^n) \otimes \Lambda^i(\mathbb{R}^n)$ after defining some necessary linear maps.

Definition 17 *Given $\alpha \otimes \beta \in \Lambda^j \otimes \Lambda^k$ define the maps*

$$*_l : \Lambda^j \otimes \Lambda^k \longrightarrow \Lambda^{n-j} \otimes \Lambda^k \tag{3.20}$$

$$*_r : \Lambda^j \otimes \Lambda^k \longrightarrow \Lambda^j \otimes \Lambda^{n-k} \tag{3.21}$$

by the rules

$$*_l (\alpha \otimes \beta) = (*\alpha) \otimes \beta \tag{3.22}$$

and

$$*_r (\alpha \otimes \beta) = \alpha \otimes (*\beta) \tag{3.23}$$

and extend by linearity.

Definition 18 *Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\Lambda^1(\mathbb{R}^n)$. We define a homomorphism of $SO(n)$ modules*

$$u : \Lambda^j \otimes \Lambda^k \longrightarrow \Lambda^{j+1} \otimes \Lambda^{k+1} \tag{3.24}$$

by linearly extending the map on monomials given by

$$u(\alpha \otimes \beta) = \sum_{i=i}^{n} \alpha \wedge e_i \otimes e_i \wedge \beta. \tag{3.25}$$

Here u is of course independent of the choice of basis made.

Definition 19 *Keeping the notation as before, we define further homomorphisms of $SO(n)$ modules*

$$l : \Lambda^j \otimes \Lambda^k \longrightarrow \Lambda^{j-1} \otimes \Lambda^{k-1} \tag{3.26}$$

$$s_L : \Lambda^j \otimes \Lambda^k \longrightarrow \Lambda^{j+1} \otimes \Lambda^{k-1} \tag{3.27}$$

$$s_R : \Lambda^j \otimes \Lambda^k \longrightarrow \Lambda^{j-1} \otimes \Lambda^{k+1} \tag{3.28}$$

by the maps on monomials given by

$$s_R(\alpha \otimes \beta) = *_l \circ *_r(u(*_l \circ *_r(\alpha \otimes \beta))) \tag{3.29}$$

$$s_L(\alpha \otimes \beta) = *_r(u(*_r(\alpha \otimes \beta))) \tag{3.30}$$

$$l(\alpha \otimes \beta) = *_l(u(*_l(\alpha \otimes \beta))) \tag{3.31}$$

With this said, we define one last class of $SO(n)$ representations which will be needed in the decompositions that follow.

Definition 20

$$W^i(\mathbb{R}^n) \equiv C(\Lambda^2(\mathbb{R}^n), \Lambda^i(\mathbb{R}^n)) \tag{3.32}$$

with $2 \leq i \leq n-2$.

We now are in a position to completely decompose the curvature summands which will occur for our choices of principal bundles $P^{\pm}$ to be made in 3.43 and 3.44.

Proposition 21

$$\Lambda^2(\mathbb{R}^8) \otimes \Lambda^2(\mathbb{R}^8) \cong \Lambda^0(\mathbb{R}^8) \oplus Y_1(\mathbb{R}^8) \oplus \Lambda^2(\mathbb{R}^8) \oplus Y_3(\mathbb{R}^8) \oplus \Lambda^4(\mathbb{R}^8) \oplus W^2(\mathbb{R}^8) \tag{3.33}$$

Proof: We begin by noting that no two of the above $SO(8)$-modules appearing as proposed summands are isomorphic; this is seen easily from the preceding discussion of their highest weights.

Now consider the map $l \circ u : \Lambda^1 \otimes \Lambda^1 \longrightarrow \Lambda^1 \otimes \Lambda^1$. The trivial computation on any (and hence all) element(s) of the three irreducible subspaces shows that this is an isomorphism of $SO(8)$ modules with eigenvalue -6 on both the Λ^2 and Y_1 subspaces and eigenvalue -14 on the Λ^0 subspace.

Likewise the map $s_R \circ s_L : \Lambda^1 \otimes \Lambda^3 \longrightarrow \Lambda^1 \otimes \Lambda^3$ is an isomorphism with eigenvalue -6 on Λ^4, and -2 on both Y_3 and Λ^2.

Now this means that both l and s_R are surjections with W^2 in both kernels. However it is then checked that the sum of the dimensions of the modules $\Lambda^1 \otimes \Lambda^3$, $\Lambda^2 \otimes \Lambda^2$, and W^2 yields

$$Dim(\Lambda^1 \otimes \Lambda^1) + Dim(\Lambda^1 \otimes \Lambda^3) + Dim(W^2) = 64 + 448 + 300 = 812 \tag{3.34}$$

whereas $Dim(\Lambda^2 \otimes \Lambda^2) = 784$, so the map

$$l \oplus s_R \oplus \Pi_{W^2} : \Lambda^2 \otimes \Lambda^2 \longrightarrow (\Lambda^1 \otimes \Lambda^1) \oplus (\Lambda^1 \otimes \Lambda^3) \oplus W^2 \tag{3.35}$$

cannot be surjective. The only possibility is then that there is a repeated summand (of dimension 28) in the range of 3.35 which appears with lower multiplicity in the domain. This must then be Λ^2 as, in addition to having the correct dimension, it is the only isotypic summand of non-simple multiplicity and the claim is established.

Proposition 22

$$\Lambda^2(\mathbb{R}^n) \otimes \Lambda_4^{\pm}(\mathbb{R}^n) \cong \Lambda^2(\mathbb{R}^n) \oplus Y_3(\mathbb{R}^n) \oplus \Lambda_{\pm}^4(\mathbb{R}^n) \oplus W_{\pm}^4(\mathbb{R}^n) \tag{3.36}$$

This is entirely analogous to the above claim and is proved in the same fashion.

3.2 Geometric Constructions

Consider a closed oriented Riemannian manifold $(X^8, g(\cdot,\cdot))$ of dimension 8. Our objects of study are certain connections on principal G-bundles

$$\begin{array}{ccc} P_G & \hookleftarrow & G \\ \pi \downarrow & & \\ X^8 & & \end{array} \tag{3.37}$$

functorially associated to our space (X^8, g_{ij}).

We will need in what follows to work with quotients of classical compact reductive lie groups. Specifically we recall that the central subgroup $Z(Spin(4n))$ is a group of order 4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for all n. We consider the quotient homomorphism

$$\pi : SO(4n) \longrightarrow Spin(4n)/Z(Spin(4n)) \cong PSO(4n) \cong SO(4n)/-1. \tag{3.38}$$

Now it is well known that the symmetry $\lambda : D_4 \longrightarrow D_4$ of order three of the Dynkin diagram D_4 in Figure 2.1 extends to the group $Spin(8)$, though not $SO(8)$. As such, it restricts to $Z(Spin(4n))$ acting as automorphisms of the central subgroup. In fact we can see that

$$OuterAut(Spin(8))/InnerAut(Spin(8)) \cong Aut(Z(Spin(4n))) = \mathbb{D}^3 \tag{3.39}$$

where $\mathbb{D}^3$ is meant to denote the dihedral group of order 6. We see then that we are at liberty to extend the triality rotation to the factor group.

$$\hat{\lambda} : PSO(8) \longrightarrow PSO(8). \tag{3.40}$$

Now consider $Z(SU(n))$. We see clearly that since all central elements are diagonal, that any element $g \in Z(SU(n)) \cong \mathbb{Z}/n\mathbb{Z}$ is realized as the n x n identity matrix multiplied by an n^{th} root of unity. Thus we have the fact that $-1 \in U(n)$ lives inside of SU(n) iff n is even. We will consdier here the case of $\widehat{SU}(n) = SU(n)/-1$ for n=8.

Let us compose the general homomorphisms $\rho : Spin(n) \longrightarrow SO(n)$ and $\iota : SO(n) \hookrightarrow SU(n)$ with the triality rotation $\lambda : Spin(8) \longrightarrow Spin(8)$ peculiar to our case. Thus we get three distinct and inequivalent maps

$$\iota \circ \rho \circ \lambda^m : Spin(8) \longrightarrow SU(8) \tag{3.41}$$

where m =0,1 or -1. Now as the natural inclusion sends $-1 \in SO(n)$ to $-1 \in SU(n)$ and the three powers of the triality rotation send in turn the three central elements of order 2 in $Z(Spin(8))$ to $-1 \in SO(8)$, we can see that we can construct three distinct homomorphisms

$$\hat{\iota} \circ \hat{\lambda}^m \circ \pi : SO(8) \longrightarrow \widehat{SU}(8). \tag{3.42}$$

Now let $\hat{P}^m$ denote the 3 principal $\widehat{SU}(8)$ bundles

$$\hat{P}^m = P^m_{\widehat{SU}(8)} = P_{Fr} \times_{\hat{\iota}\circ\hat{\lambda}^m\circ\pi} \widehat{SU}(8). \tag{3.43}$$

We will not be interested here in the case m = 0 and will prefer to concentrate on the other two possibilities. For either of these two possibilities we further construct the bundles

$$P^\pm = P^\pm_{\hat{U}(8)} = P^\pm_{\widehat{SU}(8)} \times_\iota U(8)/(\pm 1) \tag{3.44}$$

where now $\iota : \widehat{SU}(8) \longrightarrow \hat{U}(8) = U(8)/(\pm 1)$ is the obvious inclusion homomorphism. These can also be thought of as associated directly to P_{Fr} via the complexified projective semi-spin representations

$$P\Delta^\pm_{\mathbb{C}} : SO(8) \longrightarrow U(8)/\pm 1 \tag{3.45}$$

Corollary 23 *(of 2.16)*

$$Ad(P^\pm) \cong \Lambda^0(T^*X) \oplus \Lambda^2(T^*X) \oplus \Lambda^4_\pm(T^*X) \tag{3.46}$$

with the containments

$$Ad(P^\pm_{PSO(8)}) \subset Ad(P^\pm_{\widehat{SU}(8)}) \subset Ad(P^\pm) \tag{3.47}$$

given by the chain

$$\Lambda^2 \subset \Lambda^2 \oplus \Lambda^4_\pm \subset \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4_\pm. \tag{3.48}$$

Proof:From 2.16 it is clear that on a spin-manifold 3.46 holds. To see it is true in general is just the statement that the property is local in nature for our choice of bundles.

A straight dimension count establishes the chain of inclusions as all three summands are of different dimensions (28, 35, 1). This can be seen more constructively by noticing that $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{su}(n)$ are both real forms of $\mathfrak{sl}(n,\mathbb{C})$ containing $\mathfrak{so}(n)$ via the inclusions induced by their defining representations. in this case we have an isomorphism of $\mathfrak{so}(n)$-modules

$$\mathfrak{su}(n) \cong \mathfrak{sl}(n,\mathbb{R}) \cong \Lambda^2(\mathbb{R}^n) \oplus S_0^2(\mathbb{R}^n). \tag{3.49}$$

In the case of dimension 8 however we see that $\Lambda^2(\mathbb{R}^n)$ is stable under triality whereas $S_0^2(\mathbb{R}^n)$ gets rotated into $\Lambda^4_\pm$ by the powers of the triality rotation.

Constructs 24 *For $\omega \in \Omega^i(T^*X)$ we define*

$$\not{d}(\omega) = \Pi_{Y_i} \circ \nabla^{A_{L-C}}(\omega). \tag{3.50}$$

Likewise, if E_ρ is a vector bundle associated to P_G with connection A over X and $\Upsilon \in \Omega^i(T^*X, E)$, we define $\not{d}^A\Upsilon \in \Gamma^\infty(Y_i \otimes E)$ by the rule on simple sections

$$\not{d}^A(v \otimes \xi) = \not{d}(v) \otimes \xi + \Pi_{Y_i}(v) \otimes \nabla^A(\xi)) \tag{3.51}$$

and extending by linearity. Lastly we define $\check{d}^A : \Omega^i \otimes E \longrightarrow \Omega^1 \otimes \Omega^i \otimes E$ as $\check{d}^A = d_A^* + \not{d}^A$

Note that with our definitions we get a decomposition

$$\nabla(\omega) = d\omega + d^*\omega + \not{d}\omega \tag{3.52}$$

as $d = \Pi_{\Omega^{i+1}} \circ \nabla$ and $d^* = \Pi_{\Omega^{i-1}} \circ \nabla$ (see for example [5]).

Definition 25 *If*

$$\iota : \mathcal{A}(P_{Fr}) \hookrightarrow \mathcal{A}(P_G) \tag{3.53}$$

is the inclusion of 2.26, then we define the extended Levi-Civita connection to be

$$\tilde{A}_{L-C} = \iota(A_{L-C}) \in \mathcal{A}_{P_G} \tag{3.54}$$

Definition 26 *Let $\rho : SO(n) \longrightarrow G$ be a representation associating a Principal G-bundle $P_G = P_{Fr} \times_\rho G$ to the structure bundle of a Reimannian Manifold (X^n, g_{ij}). We define the extended torsion $\tau(A) \in \Omega^1(Ad(P_G))$ of a connection $A \in \mathcal{A}(P_G)$ to be*

$$\tau(A) = A - \tilde{A}_{L-C} \tag{3.55}$$

where $\tilde{A}_{L-C}$ is the extended Levi-Civita connection of P_G.

Now we define an Ad valued Symmetric expression.

Construct 27 *We define $T(A, g_{ij}) \in \Gamma^\infty(S^2 \otimes Ad(P^\pm))$ to be the section*

$$T(A, g_{ij}) = \check{d}^A(\tau(A)) \tag{3.56}$$

Without major alterations, this could be a more general ansatz of the form

$$T(A, g_{ij}) = T_{\hat{A}_{L-C}} + \check{d}^A(\tau(A)) + \eta(\tau(A)) \tag{3.57}$$

similar to the corresponding expression 2.31 for the curvature, where $T_{\hat{A}_{L-C}} \in \Gamma^\infty(S^2 \otimes Ad(P^\pm))$ would be a well defined tensor field comming from a first variation of the metric and η, some (0^{th} order) $S^2 \otimes Ad(P^\pm)$-valued expression in the torsion which vanishes upon linearization. For the purposes that we intend, we make the simplest assumption as it suffices to prove our main result. We however feel strongly that reasoning coming from variational/physical principals should produce a more meaningful (albeit more complicated) expression.

Note: It may at first seem that it would be difficult to construct such a tensor with a non-trivial quadratic expression for η valued in $S^2 \otimes Ad(P^\pm)$ analogous to the corresponding summand $[\tau, \tau]$ appearing in the curvature formula 2.31 given the symmetric nature of the first factor and anti-symmetry of the Lie bracket. It should be remembered however that the Lie algebra $\mathfrak{u}(n)$ carries a natural $U(n)$-equivariant *symmetric* product giving it the structure of a Jordan algebra, in addition to its anti-symmetric bracket operation. This can be seen in two ways.

In the first case, we note that just as the commutator of two skew-hermitian matricies is itself skew-hermitian, the anti-commutator of two hermitian matricies remains hermitian. Now since multiplication by $i \in \mathbb{C}$ maps hermitian matricies to skew-hermitian ones, it can be seen that both products are well defined on $\mathfrak{u}(n)$.

In the second case, we note that the $\mathfrak{su}(n)/\mathfrak{u}(n)$ series are peculiar in that if the isotypic summand of the trivial representation occurs with multiplicity greater than 1 in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ with $\mathfrak{g}$ simple, then $\mathfrak{g}$ must belong to the A_n series and the multiplicity must be 2. One of these invariants of course corresponds to the Lie bracket and thus determines a subspace $\mathfrak{su}(n) \subset \Lambda^2(\mathfrak{su}(n)) \subset \mathfrak{su}(n) \otimes \mathfrak{su}(n)$. The second invariant corresponds to the Jordan algebra structure alluded to above and determines a second copy $\mathfrak{su}(n) \subset S^2(\mathfrak{su}(n)) \subset \mathfrak{su}(n) \otimes \mathfrak{su}(n)$.

With this Jordan product $A * B = \frac{AB+BA}{2}$, we may then define a map

$$\xi_1 : (\Lambda^1(\mathbb{R}^{2n})\otimes S^\pm\otimes S^\pm)\otimes(\Lambda^1(\mathbb{R}^{2n})\otimes S^\pm\otimes S^\pm) \longrightarrow S^2(\mathbb{R}^{2n})\otimes S^\pm\otimes S^\pm \quad (3.58)$$

by defining $\xi_1((\mu \otimes A) \otimes (\nu \otimes B)) = (\mu \otimes \nu + \nu \otimes \mu) \otimes A * B$. Similarly we could also attempt to construct a more general expression for T_A directly from the curvature. For example $T_A = \xi_2(F_A)$ where

$$\xi_2 : (\Lambda^2(\mathbb{R}^{2n})\otimes S^\pm\otimes S^\pm)\otimes(\Lambda^2(\mathbb{R}^{2n})\otimes S^\pm\otimes S^\pm) \longrightarrow S^2(\mathbb{R}^{2n})\otimes S^\pm\otimes S^\pm \quad (3.59)$$

via the rule $\xi_2((\omega \otimes A) \otimes (\psi \otimes B)) = \Pi_{S^2}^{\Lambda^2\otimes\Lambda^2}(\omega \otimes \psi) \otimes A * B$. We will not pursue this further here.

Chapter 4

Equations

We wish to discuss the properties of the equations that we will use to define our moduli space. The following is a useful result of Atiyah and Singer.

Theorem 28 *(Atiyah-Singer) Let E_1 and E_2 be vector bundles associated to the structure bundle P_{F_r} of (X^{2n}, g_{ij}) with*

$$\mathcal{D}_j : \Gamma^\infty(E_1) \longrightarrow \Gamma^\infty(E_2) \quad j = 1, 2 \tag{4.1}$$

a pair of elliptic operators whose symbol class is likewise associated to the P_{F_r}-structure. Then $Index(\mathcal{D}_1) = Index(\mathcal{D}_2)$.

See [4] pp. 557-559 for details.

Theorem 29 *Consider any elliptic differential operator*

$$\mathcal{D} : \Gamma(E_1) \longrightarrow \Gamma(E_2) \tag{4.2}$$

over a Riemannian 8-manifold (X^8, g_{ij}) where E_1 is associated via the representation $\Lambda^1 \oplus Y_2 \oplus \Lambda^3 \oplus Y_4^{\pm}$ and E_2 is associated via $Y_1 \oplus \Lambda^2 \oplus Y_3 \oplus \Lambda^4_{\pm}$ and whose symbol is associated to the Riemannian structure as above. Then

$$Index(\mathcal{D}) = \pm \hat{A}(X^8) Ch(\Lambda^3(S^{\pm}))[X^8] \tag{4.3}$$

$$= \frac{\pm 9\chi(X^8) - \sigma(X^8)}{2} + 1024\hat{A}(X^8).$$

Note: The formula gives an integral result even if $w_2(X^8) \neq 0$ since

$$1024\hat{A}(X^8) = 4A(X^8) \tag{4.4}$$

where the A-genus is defined by $A(X^{4n}) = 16^n\hat{A}(X^{4n})$ and is always an integer.

Proof: By Theorem 28 above, the index of $\mathcal{D}$ is equivalent to the index of the twisted dirac operator

$$\not{\partial}_A \colon S^{\pm} \otimes \Lambda^3(S^{\mp}) \longrightarrow S^{\mp} \otimes \Lambda^3(S^{\mp}). \tag{4.5}$$

We recall that in dimension 8 we have the formulas (cf. e.g., [15]):

$$\hat{A}(X^8) = 1 - P_1(X^8)/24 + (7P_1^2 - 4P_2(X^8))/45 \cdot 128 \tag{4.6}$$

$$L(X^8) = 1 + P_1(X^8)/3 + (-P_1^2 + 7P_2(X^8))/45 \tag{4.7}$$

$$ch(E^r_{\mathbb{C}}) = r + c_1 + (c_1^2 - 2c_2)/2 + (c_1^3 - 3c_1c_2 + 3c_3)/6 \tag{4.8}$$

$$+ (c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4)/24 \tag{4.9}$$

$$P_i(V_{\mathbb{R}}) = (-1)^i c_{2i}(V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \tag{4.10}$$

hence,

$$ch(V^r_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})) = r + P_1 + (P_1^2 - 2P_2)/12 \tag{4.11}$$

$$P_1^2(T) = 4\sigma(X^8) + 7 \cdot 128\hat{A}(X^8) \tag{4.12}$$

$$P_2(T) = 7\sigma(X^8) + 128\hat{A}(X^8) \tag{4.13}$$

Recall that the relevant (4 and 8 dimensional) cohomology of $BSO(8)$ is generated by the universal Poyntriagn and Euler Classes P_1, P_2, $\chi \in H^*(BSO(8), \mathbb{Z})$. Now given a homomorphisim between Lie groups $\mu : G \longrightarrow H$, we get an induced map of classifying spaces

$$\tilde{\mu} : BG \longrightarrow BH. \tag{4.14}$$

In what follows we will make use of the classifying space maps induced by

$$\rho : Spin(8) \longrightarrow SO(8) \qquad \lambda : Spin(8) \longrightarrow Spin(8). \tag{4.15}$$

We are interested in understanding the characteristic classes of the semi-spin bundles in terms of the characteristic classes of the tangent bundle of our 8-manifold X^8. Let us assume for the moment that X is spin (although only products of representation which descend to SO(8) are needed).

According to [11] we have characteristic classes

$$Q_1 \in H^4(BSpin(8), \mathbb{Z}) \qquad Q_2, Y \in H^8(BSpin(8), \mathbb{Z}) \tag{4.16}$$

generating the Cohomology in these dimensions which have the following relationship to the more familiar Poyntriagn and Euler classes:

$$\tilde{\rho}^*(P_1) = 2Q_1 \tag{4.17}$$

$$\tilde{\rho}^*(P_2) = 2Q_2 + Q_1^2 \tag{4.18}$$

$$\tilde{\rho}^*(\chi) = 2Y - Q_2 \tag{4.19}$$

$$\tilde{\lambda}^*(Q_1) = Q_1 \tag{4.20}$$

$$\tilde{\lambda}^*(Q_2) = 3Y - 2Q_2 \tag{4.21}$$

$$\tilde{\lambda}^*(Y) = Y - Q_2 \tag{4.22}$$

We then compute the effect of the triality rotation on the pull back of $P_{1,2}$ and χ.

$$2Q_1 \xrightarrow{\tilde{\lambda}^*} 2Q_1 \tag{4.23}$$

$$2Q_2 + Q_1^2 \xrightarrow{\tilde{\lambda}^*} 6Y - 4Q_2 + Q_1^2 \xrightarrow{\tilde{\lambda}^*} -6Y + 2Q_2 + Q_1^2 \tag{4.24}$$

$$2Y - Q_2 \xrightarrow{\tilde{\lambda}^*} -Y \xrightarrow{\tilde{\lambda}^*} -Y + Q_2 \tag{4.25}$$

giving us:

$$P_1(S^+) = P_1(S^-) = P_1(T) \tag{4.26}$$

$$P_2(S^+) = 3\chi(T) - 1/2P_2(T) + 3/8P_1^2(T) \tag{4.27}$$

$$P_2(S^-) = -3\chi(T) - 1/2P_2(T) + 3/8P_1^2(T) \tag{4.28}$$

We compute leaving the computation in 'expanded' form so that it can be checked more easily. By the formula for the index of a twisted (positive chirality) Dirac operator (cf. e.g. [15] pp. 256) we have

$$Index(Dirac \otimes \Lambda^3(S^\pm)) = \hat{A}(X^8)ch(S^\mp \otimes T \ominus S^\pm)[X^8] \tag{4.29}$$

$$= \hat{A}(X^8)(ch(S^\mp)ch(T) - ch(S^\pm))[X^8] \tag{4.30}$$

$$= (1 - \frac{P_1(T)}{24} + \frac{(7P_1^2 - 4P_2(T))}{5760})$$

$$((8+P_1(S^\mp)+\frac{(P_1^2(S^\mp)-2P_2(S^\mp))}{12})(8+P_1(T)+\frac{(P_1^2(T)-2P_2(T))}{12})$$

$$-(8+P_1(S^\pm)+\frac{(P_1^2(S^\pm)-2P_2(S^\pm))}{12}))[X^8] \quad (4.31)$$

$$=(1-\frac{P_1(T)}{24}+\frac{(7P_1^2-4P_2(T))}{5760})$$

$$(64+8P_1(S^\mp)+8P_1(T)+P_1(S^\mp)P_1(T)$$

$$+\frac{8(P_1^2(S^\mp)-2P_2(S^\mp)+P_1^2(T)-2P_2(T))}{12}$$

$$-8-P_1(S^\pm)-\frac{(P_1^2(S^\pm)-2P_2(S^\pm))}{12})[X^8] \quad (4.32)$$

$$=(1-\frac{P_1(T)}{24}+\frac{(7P_1^2-4P_2(T))}{5760})(56+15P_1(T)$$

$$+\frac{27P_1^2(T)+2P_2(S^\pm)-16P_2(S^\mp)-16P_2(T)}{12})[X^8] \quad (4.33)$$

$$=(\frac{56(7P_1^2(T)-4P_2(T))}{5760}-\frac{15P_1^2(T)}{24}$$

$$+\frac{27P_1^2(T)+2P_2(S^\pm)-16P_2(S^\mp)-16P_2(T)}{12})[X^8] \quad (4.34)$$

$$=(\frac{56(7P_1^2(T)-4P_2(T))-15\cdot 240P_1^2(T)}{5760}$$

$$+\frac{480(27P_1^2(T)+2P_2(S^\pm)-16P_2(S^\mp)-16P_2(T))}{5760})[X^8] \quad (4.35)$$

$$=(\frac{(56\cdot 7-15\cdot 240+27\cdot 480)P_1^2(T)}{5760}$$

$$+\frac{(-4\cdot 56-16\cdot 480)P_2(T)+960P_2(S^\pm)+(-16\cdot 480)P_2(S^\mp)}{5760})[X^8] \quad (4.36)$$

$$=\frac{1}{5760}(A\cdot P_1^2(T)+B\cdot P_2(T)+C\cdot P_2(S^\pm)+D\cdot P_2(S^\mp))[X^8] \quad (4.37)$$

with $A=9752, B=-7904, C=960, D=-7680$ so we get either one of the two sign dependent possibilities:

$$=\frac{1}{5760}((A+\frac{3}{8}(C+D))P_1^2(T)+(B-\frac{1}{2}(C+D))P_2(T)\pm 3(C-D)\chi(T)) \quad (4.38)$$

$$= \frac{1}{5760}((A + \frac{3}{8}(C + D))(4\sigma(X^8) + 7 \cdot 128\hat{A}(X^8))$$

$$+ (B - \frac{1}{2}(C + D))(7\sigma(X^8) + 128\hat{A}(X^8)) \pm 3(C - D)\chi(T)) \qquad (4.39)$$

$$= \frac{1}{5760}(4A + 7B - 2(C + D))\sigma(X^8)$$

$$+ 128(7A + B + \frac{17}{8}(C + D)\hat{A}(X^8) \pm 3(C - D)\chi(T)) \qquad (4.40)$$

which upon substitution for A,B,C and D yields:

$$Index(\mathcal{D}) = \frac{\pm 9\chi(X^8) - \sigma(X^8)}{2} + 1024\hat{A}(X^8). \qquad (4.41)$$

Corollary 30 *The index of any such operator $\mathcal{D} : \Gamma^\infty(S^\pm \otimes \Lambda^3(S^\mp)) \longrightarrow \Gamma^\infty(S^\mp \otimes \Lambda^3(S^\mp))$ as in Theorem 29 satisfies $Index(\mathcal{D}) = 9$ over S^8.*

Proof: The signature and $\hat{A}$-genus are trivial as $H^4(S^8, \mathbb{Z})$ is zero and any spin manifold X^{4n} admitting a circle action which extends to its spin-structure must have $\hat{A}(X^{4n}) = 0$ by a theorem of Atiyah and Hirzebruch. As the Euler characteristic of an even sphere gives $\chi(S^{2n}) = 2$ we have $Index(\mathcal{D}) = 9$ as claimed.

Equations 31 *Let $\Phi \equiv \Phi^0 \oplus \Phi^1$ where*

$$\Phi^0 : S^\pm \otimes \Lambda^3(S^\mp) \hookrightarrow \Lambda^1(\mathbb{R}^8) \otimes S^\pm \otimes S^\pm \qquad (4.42)$$

$$\Phi^1 : S^\mp \otimes \Lambda^3(S^\pm) \hookrightarrow \Lambda^1(\mathbb{R}^8) \otimes \Lambda^1(\mathbb{R}^8) \otimes S^\pm \otimes S^\pm \qquad (4.43)$$

are injections of $SO(8)$ modules. Then we define the complexified projective self-dual (resp. anti-self dual) Einstein-Yang-Mills-Dirac equations of Euclidean signature relative to Φ to be

$$\Pi_{Im(\Phi^0)}(\tau(A)) = \tau(A) \qquad (4.44)$$

$$\Pi_{Im(\Phi^1)}(F_A \oplus T_A) = 0 \qquad (4.45)$$

Proposition 32 *Let (X^8, g_{ij}) be an Einstein metric on an oriented 8-manifold X^8. Then $\hat{A}_{L-C}$ solves the system of Equations 31.*

Proof: As the connection $\hat{A}_{L-C}$ is torsion free we have $\tau(\hat{A}_{L-C}) = T_{\hat{A}_{L-C}} = 0$ by assumption, so the only concern is the curvature. The Riemann curvature tensor $R_{ijk}{}^{l}$ decomposes as

$$R_{ijk}{}^{l} = \Pi_{S_0^2} R_{ijk}{}^{l} \oplus \Pi_{\Lambda^0} R_{ijk}{}^{l} \oplus \Pi_{W_2} R_{ijk}{}^{l} \tag{4.46}$$

where the first two terms are the Ricci curvature tensor $R_{\mu\nu}$ and the last term is the conformally invariant Weyl curvature tensor. The Equation for a metric to satisfy the Einstein condition is equivalent to requiring that

$$\Pi_{S_0^2} R_{ijk}{}^{l} = 0 \tag{4.47}$$

or equivalently that the traceless Ricci curvature vanish identically. In our notation $S_0^2 \cong Y_1$ which is the only summand of the Riemmanian curvature which appears in our equations. Hence we are done.

Now consider

$$\mathcal{A}(P_{Fr}) \longrightarrow \Gamma^\infty(S^\pm \otimes R^\pm) \longrightarrow \mathcal{A}(P^\pm) \tag{4.48}$$

given by inclusions of $Spin(8)$-modules

$$\Lambda^1 \otimes \Lambda^2 \xrightarrow{\alpha} \Lambda^3(S^\mp) \otimes S^\pm \xrightarrow{\Phi^0} \Lambda^1 \otimes S^\pm \otimes S^\pm \tag{4.49}$$

where α is the unique orthogonal inclusion.

Definitions 33 *We will refer to an extended torsion tensor $\tau(A)$ of a connection $A \in \mathcal{A}(P^\pm)$ as admissible if it is in the image of Φ^0. We will call Φ^0 simple if its image lies in $\Lambda^1 \otimes (\Lambda^2 \oplus \Lambda^4_\pm)$ and frame compatible if $\alpha \circ \Phi^0$ is equivalent to the map induced by the canonical inclusion*

$$\Lambda^2 \hookrightarrow S^\pm \otimes S^\pm \cong \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4_\pm.$$

Lastly, we will say that Φ^1 is Lorenzian if $S_0^2 \otimes S^\pm \otimes S^\pm \subset ker(\Pi_{Im(\Phi^1)})$.

Chapter 5

Solutions

5.1 Conformal Solutions on S^8

Let $\widehat{SO}(n)$ be the group of conformal transformations of $\mathbb{R}^n$ fixing the origin. Then given a conformal structure η on a manifold X^8, we define the conformal frame bundle

$$\begin{array}{lcl} P_{\widehat{Fr}}(S^8) & \hookleftarrow & \widehat{SO}(n) \cong SO(8) \times \mathbb{R}^+ \\ \pi_1 \downarrow & & \\ \quad X^8 & & \end{array} \tag{5.1}$$

as the sub-bundle of the full linear frame bundle $P_{fr}(T^*X)$ given by

$$P_{\widehat{Fr}} = \{\{e_x^i\}_{i=1}^8 \ s.t.\ e_x^i \in T_xS^8 \text{ with } g(e_x^i, e_x^j) = s\delta^{ij} \ \forall i, j\,, [g] = \eta, \text{and } s \in \mathbb{R}^+\} \tag{5.2}$$

so that given a choice of Riemannian metric compatible with the conformal structure we have a natural fibration

$$\begin{array}{lcl} P_{\widehat{Fr}}(S^8) & \hookleftarrow & \mathbb{R}^+ \\ \pi_2 \downarrow & & \\ \quad P_{Fr} & & \end{array} \tag{5.3}$$

by considering the map which assigns to each conformal frame the unique orthonormal representative of its conformal equivalence class; dually, this fibration may be thought of as having a canonical section $s_{g_{ij}} : P_{Fr}(S^8) \longrightarrow P_{\widehat{Fr}}(S^8)$. By design, for any $\gamma \in \mathit{Conf}(S^8)$ we have a natural bundle isomorphism

$$\hat{\gamma} : P_{\widehat{Fr}} \longrightarrow P_{\widehat{Fr}} \tag{5.4}$$

covering γ. This gives us a natural action of the group of conformomorphisms on $\mathcal{A}(P_{Fr})$ by $\gamma \cdot A \equiv (\pi_2 \circ \hat{\gamma} \circ s_{g_{ij}})^*(A)$.

Theorem 34 *In the above notation, let Φ be a metric injection of $SO(8)$-modules determining equations with Φ^0 simple and frame compatible, and Φ^1 Lorentzian. Then the translates of $\hat{A}_{L-C}(S^8)$ by conformal dilations solve Equations 31.*

Proof: Let $\gamma \in Conf(S^8) \cong SO(9,1)$ be an arbitrary conformomorphim and p and λ be as in Section 2.3. Then it suffices to demonstrate that the equations are solved at an arbitrary point $q \in S^8$. We examine the behavior of the tensors $\tau(A)$, F_A, T_A for $A = (\pi_2 \circ \hat{\gamma} \circ s_{g_{ij}})^*(\hat{A}_{L-C}(S^8))$.

By assumption, Φ^0 is frame compatible so the torsion is all admissible. We could actually weaken the assumption by just requiring Φ^0 to be simple in the language of 33 as all torsion of 2.43 lives within the summand $\Omega^1 \subset \Omega^1 \otimes \Omega^2$ and this representation does not appear within $\Omega^1 \otimes \Omega^4_\pm$ by Proposition 12. We will then restrict our attention to the frame bundle as the simplicity of Φ^0 means that F_A, T_A will be valued in $\Omega^2 \otimes Ad(P_{Fr})$ and $\Gamma^\infty(S^2(T^*X) \otimes Ad(P_{Fr}))$.

Recall that the definition of curvature tells us that if $f : P_G \longrightarrow Q_G$ is a bundlemap and $A \in \mathcal{A}(Q_G)$, then $F_{f^*(A)} = f^*(F_A)$. We must then examine the effect of pulling back a connection with respect to the decompositions of Propositions 21 and 22.

Now the curvature of $A_{L-C} \in \mathcal{A}(P_{Fr})$ is all scalar curvature. This is clear because given any $x \in S^8$ we have $Stab(x) = SO(8)$ acting on S^8 by isometries. Consider then that both $\Pi_{Weyl}({R_{ijk}}^l)$ and $\Pi_{S_0^2}({R_{ijk}}^l)$ must be fixed under this action. Thus we must have $\Pi_{Weyl}({R_{ijk}}^l) = \Pi_{S_0^2}({R_{ijk}}^l) = 0$ as S_0^2 and $Weyl = C^2(\Lambda^2)$ are irreducible $SO(8)$ representations.

Given that $Ad : SO(n) \longrightarrow SO(\Lambda^2(\mathbb{R}^n))$ is an irreducible contragradiant representation for $n \neq 4$, Shur's lemma tells us that the only invariants of $\Lambda^2(\mathbb{R}^n) \otimes \Lambda^2(\mathbb{R}^n)$ are the multiples of $\sum_{i<j}(dx_i \wedge dx_j) \otimes (dx_i \wedge dx_j)$. Thus we know that the curvature is of the form

$$R = \sum_{i<j} R_{[ij][ij]}(dx_i \wedge dx_j) \otimes (dx_i \wedge dx_j) \tag{5.5}$$

where $R_{[ij][ij]}$ is a constant (independent of i, j and $p \in S^8$).

We wish then to consider the difference between the pullback of Λ^2 under the usual operation of pull-back of forms and the pull back of $Ad(P_{Fr})$. To this end we consider the natural bundle maps

$$P_{\widehat{Fr}} \overset{\iota_1}{\hookrightarrow} P_{fr} \qquad P_{\widehat{Fr}} \xrightarrow{\pi_2} P_{Fr} \overset{\iota_2}{\hookrightarrow} P_{fr} \tag{5.6}$$

where the first map corresponds to the tautological inclusion of 5.2 and the second to the composition of the projection of 5.3 with the natural inclusion of the orthonormal frame bundle within the full linear frame bundle. Then the two notions of pull-back are represented by the sequences of bundle maps

$$\begin{tikzcd}
 & & & P_{fr} \\
P_{Fr} \arrow[r, "s_{g_{ij}}"] & P_{\widehat{Fr}} \arrow[r, "\gamma^*"] & P_{\widehat{Fr}} \arrow[ur, "\iota_1"] \arrow[dr, "\pi_2"'] & \\
 & & & P_{Fr} \arrow[r, hook, "\iota_2"] & P_{fr}
\end{tikzcd} \tag{5.7}$$

where the pull back of $Ad(P_{Fr})$ under $(\iota_2 \circ \pi_2 \circ \gamma^* \circ s_{g_{ij}})^*$ differs from the usual pullback of two-forms (i.e., of $\Lambda^2(T^*X) \cong P_{fr} \times_{\Lambda^2} \Lambda^2(\mathbb{R}^n)$ via $(\iota_1 \circ \gamma^* \circ s_{g_{ij}})^*$) by the action of $\mathbb{R}^+$ in $\widehat{SO}(n)$ acting as intertwiners of the $SO(n)$ representations. This means that while the pullback of the Levi-Civita connection will not have constant curvature, none of the curvature's other summands will contribute to the curvature. As the scalar summand does not appear in the decomposition of Proposition 16, we know that the curvature of the conformal translates of $\hat{A}_{L-C}(S^8)$ must satisfy $\Pi_{Im(\Phi^1)}(F_A) = 0$.

It remains to check that $\Phi^1(T_A) = 0$. Let us assume that we are given a conformal dilation γ centered at $p \in S^n$. We want to examine the behaviour of T_A at an arbitrary point q. First let us assume that $p \neq q$, and denote by $SO(7)_{p,q} \hookrightarrow SO(8)_p \hookrightarrow SO(9)$ the inclusions of the groups stabilizing the points of the great circle joining p and q, and the point p itself. The pullback of the Levi-Civita connection under γ will not have the full $SO(9)$ symmetry of A_{L-C} but it will be invariant under $SO(7)_{p,q}$. As $\Phi^1(T_A) \in \Omega^0 \otimes Ad(P^\pm) \subset \Gamma^\infty(S^2 \otimes Ad(P^\pm))$ by the assumption that Φ^1 is Lorentzian, we can examine the value of $\Phi^1(T_A)$ at q under the action of $SO(7)_{p,q}$. We know that the extended torsion is zero and so $T_A \in \Omega^0 \otimes \Omega^2 \subset \Omega^0 \otimes Ad(P^\pm)$ and thus we must decompose $\Lambda^2(\mathbb{R}^8)$ under $SO(7)_{p,q}$. As this is given by

$$\Lambda^2(\mathbb{R}^8) \cong_{SO(7)_{p,q}} \Lambda^2(\mathbb{R}^7) \oplus \Lambda^1(\mathbb{R}^7) \tag{5.8}$$

we see that there are no 1-dimensional invariant subspaces and therefore we must have $\Phi^1(T_A) = 0$ for $q \neq \pm p$. Continuity (or the irreducibility of $\Lambda^2(\mathbb{R}^8)$) establishes the rest.

Corollary 35 *The system*

$$\Pi_{\Lambda^1}^{\Lambda^1 \otimes \Lambda^0}(\tau(A)) = \Pi_{\Lambda^3}^{\Lambda^1 \otimes \Lambda^4_\pm}(\tau(A)) = 0 \tag{5.9}$$

$$\Pi_{\Lambda^4_\pm}^{\Lambda^2 \otimes \Lambda^4_\pm}(F_A) = \Pi_{\Lambda^4_\pm}^{\Lambda^0 \otimes \Lambda^4_\pm}(T_A) - \Pi_{\Lambda^4_\pm}^{\Lambda^2 \otimes \Lambda^2}(F_A) \tag{5.10}$$

$$\Pi_{Y_3}^{\Lambda^2 \otimes \Lambda^4_\pm}(F_A) = -\Pi_{Y_3}^{\Lambda^2 \otimes \Lambda^2}(F_A) \tag{5.11}$$

$$\Pi_{\Lambda^2}^{\Lambda^2 \otimes \Lambda^4_\pm}(F_A) = \Pi_{\Lambda^2}^{\Lambda^0 \otimes \Lambda^2}(T_A) - \Pi_{\Lambda^2}^{\Lambda^2 \otimes \Lambda^2}(F_A) \tag{5.12}$$

$$\Pi_{Y_1}(F_A) = 0 \tag{5.13}$$

is an elliptic system of equations with index 9 on S^8 and $SO(9,1)/SO(9) \hookrightarrow \mathcal{M}(S^8)$. Here Π_V^U denotes the projection $\Pi_V^U : U \longrightarrow V$ for (reducible) representations $U \cong V \oplus W$. As all such V's are unique in the above, these are well defined without further specification.

Proof: This system corresponds to the 5^{th} set of constants A.5 in the appendix. The determinant of the symbol matrix μ (in the basis used by the program) is computed over the integers to be $Det(\mu) = -1 \cdot 2^{512} \cdot 3^{99} \cdot 5^{21} \cdot 7^8 \neq 0$ so the equations are elliptic. This set of equations fits the criteria of the theorem by implicitly determining an elliptic choice of $\Phi = \Phi^0 + \Phi^1$ which is frame-compatible, special and Lorentzian so we are done.

5.2 Abstract Solution to manifestly elliptic equations on S^8

We give below similar equations whose elliptic character is more transparent.

By Proposition 14, $R^\pm$ is uniquely contained as a sub-representation of $\Lambda^1 \otimes S^\pm$; we will denote this inclusion by ι_0. There is therfore a unique inclusion map

$$\iota_1 : R^\pm \otimes S^\pm \hookrightarrow \Lambda^1 \otimes S^\pm \otimes S^\pm \tag{5.14}$$

extending ι_0 which acts as the identity on the last $S^{\pm}$ factor of the domain and range. Recalling that $Ad(P^{\pm}) \cong S^{\pm} \otimes S^{\pm}$, we define Φ^0 by the map ι_1 above.

To define the Φ^1 injection of $R^{\pm} \otimes S^{\mp}$ into $\Lambda^1 \otimes \Lambda^1 \otimes S^{\pm} \otimes S^{\pm}$ we use the ι_0 inclusion to include $R^{\pm}$ into the 2^{nd} and 3^{rd} factors while taking the adjoint of the clifford multiplication map of 2.20 to include $S^{\mp}$ into the tensor product of the 1^{st} and 4^{th} factors.

With these definitions it is then clear that if we retain our choice of T_A, that the linearization of these equations is precisely the the $\pm$-chirality Dirac operator twisted by the $R^{\pm}$ coeffificient bundle. The extended Levi-Civita connection $\hat{A}_{L-C}$ still gives us a solution by Proposition 32.

Chapter 6

Further Remarks

We collect here some remarks intended to put the previous chapters in some kind of context. These remarks will often be without proof and should be considered accordingly.

6.1 Amplification of Facts From Representation Theory

Proposition 36 *For $0 \leq i \leq j \leq \frac{n}{2}$, there exist a sequence of maps*

$$0 \longrightarrow C(\Lambda^i(\mathbb{R}^n), \Lambda^j(\mathbb{R}^n)) \hookrightarrow \Lambda^i(\mathbb{R}^n) \otimes \Lambda^j(\mathbb{R}^n) \tag{6.1}$$

$$\longrightarrow \Lambda^{i-1} \otimes (\Lambda^{j-1}(\mathbb{R}^n) \oplus \Lambda^{j+1}(\mathbb{R}^n)) \longrightarrow \Lambda^{i-2}(\mathbb{R}^n) \otimes \Lambda^j(\mathbb{R}^n) \longrightarrow 0$$

which is exact as a sequence of $SO(n)$-modules for any n.

Definition 37

$$S_0^i(\mathbb{R}^n) \equiv S^i(\mathbb{R}^n) \ominus S^{i-2}(\mathbb{R}^n) \tag{6.2}$$

or equivalently $S_0^i(\mathbb{R}^n) \cong C^i(\mathbb{R}^n)$.

Proposition 38 *For any n, there exist a sequence of maps such that*

$$0 \longrightarrow C(\Lambda^i, S^{\pm}) \hookrightarrow \Lambda^i \otimes S^{\pm} \longrightarrow \Lambda^{i-1} \otimes S^{\mp} \longrightarrow 0 \quad \text{for } i \leq n \tag{6.3}$$

is exact as a sequence of $SO(2n)$ representations.

Proposition 39 *For any n, there exist a sequence of maps such that*

$$0 \longrightarrow C(S_0^i, S^\pm) \hookrightarrow S^i \otimes S^\pm \longrightarrow S^{i-1} \otimes S^\mp \longrightarrow 0 \tag{6.4}$$

is exact as a sequence of $SO(2n)$ representations.

Corollary 40 *1. $Y_i(\mathbb{R}^n)$ is irreducible of dimension $\binom{n}{i} - \binom{n}{i-1} - \binom{n}{i+1}$ except in the case of middle dimension where the decomposition into irreducibles $Y_n(\mathbb{R}^{2n}) \cong Y_n^+(\mathbb{R}^{2n}) \oplus Y_n^-(\mathbb{R}^{2n})$.*

6.2 Proposed Extension of Self-Dual Equations to Even Dimensions

Note: Dimensions 4n and 4n+2 have a slightly different character to them in that in dimension 4n, the spin representations are self-conjugate whether they are quaternionic or real whereas in 4n+2 they are complex and not self conjugate. We will therefore simplify and discuss extension for 4n and leave the details of 4n+2 to the interested reader who should have no difficulty.

In the preceding chapters we have used the representations $\Lambda^3(S^\pm)$ to extend the self dual equations to dimension 8. A closer look shows that an attempt to extend our equations to other dimensions via the third exterior power of the semi-spin representations does not work. The dependence on triality is reason enough not to find this too surprising.

We propose that the correct picture should come from the isomorphism $\Lambda^3(S^\pm) = C(T, S^\pm) \equiv R^\pm$ and using $R^\pm$ as the auxiliary bundle with which to twist.

Proposition 41 *For $n \in 2\mathbb{Z}$ we have*

$$S^\pm \otimes R^\pm \cong \oplus_{i=1}^{n/2}(\Lambda^{2i-1}(\mathbb{R}^{2n}) \oplus Y_{2i}(\mathbb{R}^{2n})) \ominus Y_n^\mp(\mathbb{R}^{2n}) \tag{6.5}$$

as an equivalence of $SO(2n)$ representations.

Proof:

$$S^\pm \otimes R^\pm \cong S^\pm \otimes (S^\pm \otimes T \ominus S^\mp) \tag{6.6}$$

$$\cong (\oplus_{i=0}^{n/2}\Lambda^{2i}(\mathbb{R}^{2n}) \otimes T) \ominus (\oplus_{j=1}^{n/2}\Lambda^{2i-1}(\mathbb{R}^{2n})) \tag{6.7}$$

$$\cong \Lambda^1(\mathbb{R}^{2n}) \oplus Y_2(\mathbb{R}^{2n}) \oplus \Lambda^3(\mathbb{R}^{2n}) \oplus \ldots \oplus \Lambda^{n-1}(\mathbb{R}^{2n}) \oplus Y_n^\pm(\mathbb{R}^{2n}) \tag{6.8}$$

Likewise:

Proposition 42 *For $n \in 2\mathbb{Z}$ we have*

$$S^{\mp} \otimes R^{\pm} \cong \oplus_{i=1}^{n/2}(Y_{2i-1}(\mathbb{R}^{2n}) \oplus \Lambda^{2i}(\mathbb{R}^{2n})) \ominus \Lambda^{n}_{\mp}(\mathbb{R}^{2n}) \tag{6.9}$$

as an equivalence of $SO(2n)$ representations.

Proof:

$$S^{\mp} \otimes R^{\pm} \cong S^{\mp} \otimes (S^{\pm} \otimes T \ominus S^{\mp}) \tag{6.10}$$

$$\cong (\oplus_{i=0}^{n/2}\Lambda^{2i-1}(\mathbb{R}^{2n}) \otimes T) \ominus (\oplus_{j=1}^{n/2}\Lambda^{2i}(\mathbb{R}^{2n})) \tag{6.11}$$

$$\cong Y_1(\mathbb{R}^{2n}) \oplus \Lambda^2(\mathbb{R}^{2n}) \oplus Y_3(\mathbb{R}^{2n}) \oplus \dots \oplus Y_{n-1}(\mathbb{R}^{2n}) \oplus \Lambda^{n}_{\pm}(\mathbb{R}^{2n}) \tag{6.12}$$

In these dimensions (those divisible by 4) the complexified projective spin representations are of the form

$$P\Delta^{\pm}_{\mathbb{C}} : SO(2n) \longrightarrow U(2^{2n-1})/\{\pm Id\}. \tag{6.13}$$

The Lie algebras $\mathfrak{u}^{\pm}(2^{2n-1})$ decompose under $SO(2n)$ into irreducible representations as:

$$\mathfrak{u}^{\pm}(2^{2n-1}) \cong \oplus_{i=0}^{n/2}\Lambda^{2i}(\mathbb{R}^{2n}) \ominus \Lambda^{n}_{\mp}(\mathbb{R}^{2n}). \tag{6.14}$$

We see by Proposition 36 that the equations

Equations 43

$$\Pi^{\Lambda^1\otimes\Lambda^0}_{\Lambda^1}(\tau(A)) = 0 \tag{6.15}$$

$$\Pi^{\Lambda^1\otimes\Lambda^{2i}}_{\Lambda^{2i-1}}(\tau(A)) = 0 \quad i \geq 2 \tag{6.16}$$

$$\Pi^{\Lambda^2\otimes\Lambda^2}_{Y_1}(F_A) = 0 \tag{6.17}$$

$$\Pi^{\Lambda^2\otimes\Lambda^{2i}}_{Y_{2i+1}}(F_A) = -\Pi^{\Lambda^2\otimes\Lambda^{2i+2}}_{Y_{2i+1}}(F_A) \quad i \geq 1 \tag{6.18}$$

$$\alpha\Pi^{\Lambda^2\otimes\Lambda^{2i+2}}_{\Lambda^{2i+2}}(F_A) = (\beta\Pi^{\Lambda^0\otimes\Lambda^{2i+2}}_{\Lambda^{2i+2}}(T_A) \tag{6.19}$$

$$-\gamma\Pi^{\Lambda^2\otimes\Lambda^{2i}}_{\Lambda^{2i+2}}(F_A) - \delta\Pi^{\Lambda^2\otimes\Lambda^{2i+4}}_{\Lambda^{2i+2}}(F_A)) \quad i \geq 1 \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

are well defined for connections $A \in \mathcal{A}(P_{F_r} \times_{P\Delta^{\pm}_{\mathbb{C}}} U(2^{2n-1})/\{\pm Id\})$ over (X^{2n}, g_{ij}) once the indeterminates $\alpha - \delta$ are chosen (with the proviso that one interprets Λ^n and Y_n as $\Lambda^n_{\pm}$ and $Y^{\pm}_n$ wherever they appear above).

Given any choice of constants for which these equations are elliptic, their index will be the same as that of the $\pm$-chirality dirac operators twisted by $R^{\pm}$.

Proposition 44 *The index of*

$$\not{\partial}_A \colon S^+ \otimes R^\pm \longrightarrow S^- \otimes R^\pm \tag{6.20}$$

on S^{2n} *is* $\pm(2n+1)$.

Proof: As noted before, the index of the positive chirality Dirac operator twisted by a coefficient bundle E over X^{2n} is equal to $\hat{A}(T^*(X^{2n}))ch(E)[X^{2n}]$. On the sphere S^{2n}, this is calculated by recalling that $\hat{A}(T^*(S^{2n})) = 1 \in H^0(S^{2n})$, $ch(T^*(S^{2n})) = 2n \in H^0(S^{2n})$ and $< ch(S^\pm), [S^{2n}] >= \pm 1$. This then gives us

$$Index(\not{\partial}_A) = \hat{A}(T^*(S^{2n}))ch(R^\pm)[S^{2n}] \tag{6.21}$$

$$= ch(S^\pm \otimes T^* \ominus S^\mp)[S^{2n}] = (ch(S^\pm)ch(T^*) - ch(S^\mp))[S^{2n}] \tag{6.22}$$

$$= \pm(2n+1) \tag{6.23}$$

as claimed.

Corollary 45 *The virtual dimension of the moduli space of solutions* $\mathcal{M}(S^{2n})$ *is* $2n+1$ *for elliptic choices of Equations 43 (of either chirality).*

Proof: Clear.

The question remains as to why this should have anything to do with conformal transformations. I believe that the answer should come from 'unrolling' the operator $\not{\partial}_A \colon S^+ \otimes R^\pm \longrightarrow S^- \otimes R^\pm$ into

$$\begin{array}{ccccccccccccc} & & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \cdots & \longrightarrow & Y_{n-1} & \longrightarrow & Y_n^\pm \\ & \nearrow & \oplus & \nearrow & \oplus & \nearrow & & & \nearrow & \oplus & \nearrow & \\ \Lambda^1 & \longrightarrow & \Lambda^2 & \longrightarrow & \Lambda^3 & \cdots & \longrightarrow & \Lambda^{n-1} & \longrightarrow & \Lambda^n_\pm & & \end{array} \tag{6.24}$$

by Propositions 41 and 42. While I have not (yet) made any attempt to prove that 6.24 is actually an elliptic complex, the natural symbol sequence is exact. Let us examine the first term.

As we have said before $Y_1 \cong S_0^2(T^*X)$. The natural 1^{st}-order differential operator $\delta : \Gamma^\infty(S^2(T^*)) \longrightarrow \Omega^1$ called the divergence (cf. e.g. [6] pp. 35) has as its adjoint $\delta^* \cong d^* \oplus \text{d̸} : \Omega^1 \longrightarrow \Omega^0 \oplus \Gamma^\infty(Y_1(T^*X)) \cong \Gamma^\infty(S^2(T^*X))$. Now for any $\omega \in \Omega^1(T^*X)$

$$\delta^*\omega = -\frac{1}{2}L_{\omega^\sharp}g_{ij} \tag{6.25}$$

and hence $V \in \Gamma^\infty(TX)$ is a Killing field (an infinitesimal isometry) if and only if $\delta^*(V^\flat) = 0$. It is just as easy to see that V is the infinitesimal generator of a conformal self map if and only if $(\Pi_{Y_1} \circ \delta^*)(V^\flat) = ḏ(V^\flat) = 0$. We refer to such a vector field as a conformal Killing field.

Let V be any conformal Killing field of S^{2n}. If V is an actual killing field, $V^\flat$ will be not be closed under the usual exterior derivative unless $V^\flat = 0$. This is due to the fact that by 6.25, $\delta^*(V^\flat) = 0$ implies $d^*(V^\flat) = 0$ and by Hodge theory $V = 0$ as $H^1(S^{2n}) = 0$ implies a unique harmonic 1-form.

Recall that the group of conformal transformations of S^{2n} is $SO(2n+1,1)$; exponentiation therefore gives a natural isomorphism between the Lie-algebra of conformal Killing fields under the natural bracket of vector fields, and $\mathfrak{so}(2n+1,1)$ with the space of Killing fields corresponding to the exponentiation of the sub-algebra $\mathfrak{so}(2n+1) \subset \mathfrak{so}(2n+1,1)$. An arbitrary conformal dilation ϕ_p fixing $p \in S^{2n}$ is obviously generated by an $SO(2n)$-symmetric gradient vector field $V = (df)^\sharp$. The space spanned by such fields is isomorphic to $\mathfrak{so}(2n+1,1)/\mathfrak{so}(2n+1) \cong \mathbb{R}^{2n+1}$ by the above and with any such V satisfying

$$(d \oplus ḏ)(V^\flat) = 0. \tag{6.26}$$

This is (conjecturally) the source of the 'co-homology' of 6.24 and matches the Euler characteristic of the rolled up complex computed in Proposition 44 assuming that all other 'co-homology' vanishes.

It is also worth mentioning here why the author considers this a generalization of the self-dual equations. One could of course adopt the point of view that the self dual equations are governed by an elliptic complexes based on the 'half-signature' complexes 2.37 and 2.38 whereas ours are based on complexes of the form 6.20. However, the discrepancy is not serious.

Recall that $SO(4)$ may be represented as the semi-direct product

$$SO(4) \cong S^3_{\mathbb{H}} \times_{Aut(S^3)} Aut(\mathbb{H}) \cong Spin(3) \times_{Conj} (Spin(3)/\mathbb{Z}_2). \tag{6.27}$$

Likewise

$$Spin(4) \cong Sp(1)_+ \times Sp(1)_- \tag{6.28}$$

with all irreducible representations of $Spin(4)$ being of the form $V_+ \otimes W_-$ where V and W are irreducible representations of the 2-factors. Let us denote the trivial, defining, and adjoint representations of $Sp(1)_\pm$ by $1_\pm$, $S^\pm$, and $Ad_\pm$ respectively. It is then clear that $T \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}^4_{\mathbb{C}} \cong S^+ \otimes S^-$ and that

$\mathfrak{so}(4) \cong Ad_+ \oplus Ad_-$. Now given that $S^\pm \otimes S^\pm \cong 1_\pm \oplus Ad_\pm$ we have

$$S^\pm \otimes Ad_\mp \cong (1_\pm \otimes S^\mp) \otimes (S^\pm \otimes S^\mp) \ominus (S^\pm \otimes 1_\mp). \tag{6.29}$$
$$\cong (S^\mp \otimes T^*) \ominus S^\pm \cong R^\mp.$$

For the case of the fundamental solutions to the Self-Dual-Yang-Mills equations (or ASDYME), the adjoint bundles to 1.1 are the bundles associated by $Ad_\pm$ above. Let us choose to look at the Self-Dual equations for simplicity. The fundamental deformation complex 2.42 when rolled up is equivalent to

$$\partial\!\!\!/_A \colon S^+ \otimes (S^- \otimes Ad_+) \longrightarrow S^- \otimes (S^- \otimes Ad_+). \tag{6.30}$$

But by 6.29, this is equivalent to 6.20 above. Thus we see that when the adjoint bundles to the (auxiliary) Principal bundles $P^\pm_{Sp(1)} \cong S^7$ are incorporated into the deformation complex by considering them as being associated to the structure bundle of S^4, the usual deformation complex takes on the form of 6.20.

6.3 Existence of Einstein Structures

In [6], it is stated that there are at present no known obstructions to the existence of Einstein structures (Einstein metrics upto diffeomorphism) in dimension greater than 4. Given the relation to our equations it is worth asking whether they might shed light on the question of whether all manifolds in higher dimensions carry an Einstein structure.

Recall that the Rational (oriented) Co-Bordism group in dimension 8

$$\Omega_8^{SO} \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \tag{6.31}$$

is generated by $\mathbb{CP}(2) \times \mathbb{CP}(2)$ and $\mathbb{CP}(4)$. If X^m and Y^m are oriented manifolds, then their product is represented in Ω_m^{SO} (we ignore the ring structure here) by

$$[X] \cdot [Y] \cong [X \cup Y] \cong [X \# Y] \tag{6.32}$$

where the above union is disjoint and the operation of connected sum is done with respect to the chosen orientations of X and Y.

Now the above generators carry the Pontrjagin numbers

$$< p_1^2(\mathbb{CP}(2) \times \mathbb{CP}(2)), [\mathbb{CP}(2) \times \mathbb{CP}(2)] >= 18 \tag{6.33}$$

$$< p_2(\mathbb{CP}(2) \times \mathbb{CP}(2)), [\mathbb{CP}(2) \times \mathbb{CP}(2)] >= 9 \tag{6.34}$$

$$< p_1^2(\mathbb{CP}(4)), [\mathbb{CP}(4)] >= 25 \tag{6.35}$$

$$< p_2(\mathbb{CP}(4)), [\mathbb{CP}(4)] >= 10 \tag{6.36}$$

hence by 4.6 and 4.7 we have

$$A(\mathbb{CP}(2) \times \mathbb{CP}(2)) = 256\hat{A}(\mathbb{CP}(2) \times \mathbb{CP}(2)) = 4 \tag{6.37}$$

$$A(\mathbb{CP}(4)) = 256\hat{A}(\mathbb{CP}(4)) = 6 \tag{6.38}$$

$$\sigma(\mathbb{CP}(2) \times \mathbb{CP}(2)) = 1 \tag{6.39}$$

$$\sigma(\mathbb{CP}(4)) = 1 \tag{6.40}$$

(cf. e.g. [9] pg. 346). Since the characteristic numbers above give additive homomorphisms, we get the following formula

$$\alpha_\pm \left(S^8 \# \left(\left(\#_K \left(\mathbb{CP}\left(2\right) \times \mathbb{CP}\left(2\right)\right)\right) \# \left(\#_L \left(\mathbb{CP}\left(4\right)\right)\right)\right)\right) = \alpha_\pm(X^8_{K,L}) \tag{6.41}$$

$$= \frac{9\chi(X^8_{K,L}) \mp \sigma(X^8_{K,L})}{2} \pm 4A(X^8_{K,L}) \tag{6.42}$$

$$= \frac{9\chi(X^8_{K,L}) \mp (k+l)}{2} \pm (16K + 24L) \tag{6.43}$$

$$= \frac{9\chi(X^8_{K,L}) \pm (31K + 47L)}{2} \tag{6.44}$$

$$= \frac{9(2 + 7K + 3L) \pm (31K + 47L)}{2} \tag{6.45}$$

$$= 9 + \frac{(63 \pm 31)K + (27 \pm 47)L}{2} \tag{6.46}$$

So the family

$$X^8_{5j,8j} \equiv X^8_j \equiv S^8 \# (\#_{5j}(\mathbb{CP}(2) \times \mathbb{CP}(2))) \# (\#_{8j}(\mathbb{CP}(4))) \tag{6.47}$$

satisfies

$$\alpha_-(X^8_j) = 9 \tag{6.48}$$

for all non-negative $j \in \mathbb{Z}$. This raises (in my mind) the following question. ¿If such a manifold carries an Einstein metric, what does its moduli space $\mathcal{M}_+(X_j^8)$ look like? By assumption (and Proposition 32) we know that $\mathcal{M}_+(X_j^8) \neq \{\phi\}$. Likewise, as $[X_j^8] \neq 0 \in \Omega_8^{SO}$ for $j \geq 1$, it cannot of course be a smooth oriented 9-manifold with $\partial(\mathcal{M}_+(X_j^8)) \cong X_j^8$ unless we are in the degenerate case: $j = 0$. Here we have precisely

$$\partial(\mathcal{M}_+(X_0^8)) \cong \partial(\mathcal{M}_+(S^8)) \cong \partial(B^9) \cong S^8 \tag{6.49}$$

(at least for the round Einstein metric).

6.4 Odd dimensional companion theories

In dimension 4, the theory of Self-Dual equations is closely related to theories of connections on Three-manifolds. In one such approach pioneered by Taubes (leading to the Casson invariant), the flat connections on a trivialized $SU(2)$-Principal bundle over a Homology-3-sphere (X^3, g_{ij}) are studied as the zeroes of the "vector field" $[*F_A]$ on $\mathcal{A}/\mathcal{G}$. Here the vector field has fredholm linearization making it possible to count the zero's of $[*F_A]$ (subject to a perturbation removing the possible degeneracies). Critical to this approach is the ability in dimension 3 to relate the curvature F_A to the tangent space $T_A\mathcal{A}(P_G)$ to the space of connections.

We wish to point out that a similar situation may arise here. If we are given an oriented Riemannian manifold (X^{2n+1}, g_{ij}), then we may (locally) define the twisted Dirac operator

$$\partial\!\!\!/_A \colon S \otimes R \longrightarrow S \otimes R \tag{6.50}$$

where analogously to the even dimensional situation we define $R \equiv C(T, S)$. The tensor product $T \otimes R$ is given as

$$S(\mathbb{R}^{2n+1}) \otimes R(\mathbb{R}^{2n+1}) \cong \oplus_{i=1}^{n}(\Lambda^i(\mathbb{R}^{2n+1}) \oplus Y_i(\mathbb{R}^{2n+1})) \tag{6.51}$$

whereas the complexified spin representation

$$P\Delta_{\mathbb{C}} : SO(2n+1) \longrightarrow U(2^n)/\pm 1 \tag{6.52}$$

determines principal bundles $P_{U(2^n)/\pm 1} = P_{Fr} \times_{P\Delta_{\mathbb{C}}} U(2^n)/\pm 1$ with

$$Ad(P_{U(2^n)/\pm 1}) = \Lambda^{2*}(T^*X^{2n+1}). \tag{6.53}$$

We merely wish to point out that by Proposition 36 that the Λ^i and Y_j summands which appear in 6.51 occur in both the torsion and curvature of a connection $A \in \mathcal{A}(P_{U(2^n)/\pm1})$. This means that if $\Phi^0 : S \otimes R \hookrightarrow \Lambda^1 \otimes Ad(P_{U(2^n)/\pm1})$ is an injection and $\Phi^1 : (\Lambda^2 \oplus S^2) \otimes Ad(P_{U(2^n)/\pm1}) \longrightarrow S \otimes R$ then one can take a vector field of the form $V = \Phi^0 \circ \Phi^1(F_A \oplus T_A)$ and attempt to determine its behavior as in dimension 3.

6.5 Other Choices of Bundles Governed by Other Elliptic Operators

We wish to point out here that while we have restricted our attention to the Complexified Projective Semi-spin representations, there exist many other possibilities to which this method might be applied. In addition, there is no reason to restrict ourselves to the operator

$$\not{\partial}_A : S^+ \otimes R^\pm \longrightarrow S^- \otimes R^\pm \tag{6.54}$$

if we change our bundle.

The way in which we envision this is the following. Take a pair of chiral representations $\rho^\pm$ of the structure group $SO(2n)$ of a Riemannian manifold (X^{2n}, g_{ij}). Secondly, use these representations to associate a pair of principal bundles and analyze the irreducible summands of their $\tau_A,\ F_A, and\ T_A$ tensors. Then locate and elliptic operator whose summands all appear in the above.

For example, we could take the representations

$$\rho_\pm : SO(2n) \longrightarrow SO(\Lambda^n_\pm(\mathbb{R}^{2n})) \tag{6.55}$$

and use them to form the associated principal bundle

$$P_{\rho^\pm} = P_{Fr} \times_{\rho^\pm} SO(\Lambda^n_\pm(\mathbb{R}^{2n})). \tag{6.56}$$

Now define $Q^\pm$ via

$$Q^\pm = C(\Lambda^2, S^\pm) \tag{6.57}$$

and decompose the summands of the twisted dirac operator

$$\not{\partial}_A : S^+ \otimes Q^\pm \longrightarrow S^- \otimes Q^\pm \tag{6.58}$$

for the sake of concreteness we specialize to the case of $SO(8)$ and assert that 6.58 decomposes in this case as

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & W^2 \arrow[dr] & & & \\
 & & Y_2 \arrow[ur] \arrow[dr] & \oplus & W^3 \arrow[dr] & & \\
0 \arrow[r] & \Lambda^2 \arrow[ur] \arrow[dr] & \oplus & Y_3 \arrow[ur] \arrow[dr] & \oplus & W^4_\pm \arrow[r] & 0 \\
 & & \Lambda^3 \arrow[ur] \arrow[dr] & \oplus & Y_4^\pm \arrow[ur] & & \\
 & & & \Lambda^4_\pm \arrow[ur] & & &
\end{tikzcd} \tag{6.59}$$

whereas $Ad(P_{\rho^\pm}) = \Lambda^2 \oplus W^4_\pm$. It can be deduced from Proposition 36 that

$$\Lambda^1 \otimes W^4_\pm \cong Y_4^\pm \oplus W_3 \oplus C(\Lambda^2, Y_4^\pm) \oplus C(\Lambda^3, \Lambda^4_\pm) \tag{6.60}$$

as well as the fact that $W^4_\pm \subset \Lambda^2 \otimes W^4_\pm$. Therefore the curvature and torsion tensors contain all of the necessary summands and one can begin to formulate equations just has we have done for the spin bundles. We do not pursue this here.

6.6 Phenomena Peculiar to Low Dimensions

In low dimensions (say less than 9) there are two peculiar features which distinguish possibilities for theories of connections which do not occur elsewhere:

1. The manifold (X^n, g_{ij}) has reducible, complex or repeated summands in the curvature and torsion of a connection $A \in \mathcal{A}(P_{Fr})$ which in higher dimensions would be irreducible real representations of simple multiplicity.

2. The manifold (X^n_G, g_{ij}) has an unusual G-structure determining decompositions of the exterior algebra $\Lambda^*(T^*X^n_G)$ making possible elliptic equations involving the curvature of connections on auxiliary bundles P_G.

6.6.1 Relativity + Torsion in Dimension 4

As is well known, there are *many* peculiarities of 4-dimensional topology and geometry. We wish to discuss here some of the implications of the decomposition of the full curvature tensor of an arbitrary metric connection on a Riemannian 4-manifold (X^4, g_{ij}).

From Proposition 36 we know that

$$\Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4) \cong \Lambda^0 \oplus Y_1 \oplus \Lambda^2_+ \oplus \Lambda^2_- \oplus Y_3 \oplus \Lambda^4 \oplus W^2_+ \oplus W^2_- \qquad (6.61)$$

where $S^2_0 = Y_1 \cong Y_3$ and $\Lambda^0 \cong \Lambda^4$. We note that this is the only dimension where either the Y_1 or the Λ^0 summands appear with non-simple multiplicity. To see where this might have some significance, let us recall the set up of General Relativity in dimension n=4.

Let $Met(X^n)$ be the space of C^∞ metrics on a manifold X^n and $Diff(X^n)$ denote the group of all diffeomorphisms of X^n. The tangent space to $Met(X^n)$ is $\Gamma^\infty(S^2(T^*(X^n)))$ while the Lie algebra of $Diff(X^n)$ is just the space of infinitesimal flows $Vect(X^n)$, with the Lie bracket of vector fields giving the Lie bracket in the Lie algebra.

The group $Diff(X^n)$ acts naturally on $Met(X^n)$ via $\gamma \cdot g_{\mu\nu} \equiv \gamma^*(g_{\mu\nu})$ where $\gamma \in Diff(X^n)$. At any given metric $g_{\mu\nu} \in Met(X^n)$, the induced map of tangent spaces

$$D_{g_{\mu\nu}} : Vect(X^n) \longrightarrow \Gamma^\infty(S^2(T^*(X^n))) \qquad (6.62)$$

is the adjoint of the divergence

$$\delta_{g_{\mu\nu}} : \Gamma^\infty(S^2(T^*(X^n))) \longrightarrow \Gamma^\infty(S^1(T^*(X^n))). \qquad (6.63)$$

Now assume we are modeling a physical system Σ (eg. electro-magnetism, scalar mesons) with some space of field like objects which behave well under pull back (connections, sections of vector bundles, ...) and for which we are given a local lagrange density

$$\mathcal{L}_{g_{\mu\nu}} : \Sigma \longrightarrow \mathbb{R} \qquad (6.64)$$

defining the dynamics. Then the theory of General Relativity is the statement that the spacetime metric $g_{\mu\nu}$ and matter distribution $\sigma \in \Sigma$ should satisfy

$$dS_{g_{\mu\nu}} = dT(\sigma)_{g_{\mu\nu}} \qquad (6.65)$$

on $Met(X^n)$ where

$$\mathcal{S}(g_{\mu\nu}) = \int_{X^n} s(g_{\mu\nu})\ dvol_{g_{\mu\nu}} \tag{6.66}$$

is the function

$$\mathcal{S} : Met(X^n) \longrightarrow \mathbb{R} \tag{6.67}$$

given by the integral of the scalar curvature of the Levi-Civita connection of $g_{\mu\nu}$ with respect to its natural volume form, and $\mathcal{T}(\sigma)(g_{\mu\nu}) \longrightarrow \mathbb{R}$ is determined by

$$\mathcal{T}(\sigma)(g_{\mu\nu}) \equiv \int_{X^n} \mathcal{L}_{g_{\mu\nu}}(\sigma)\ dvol_{g_{\mu\nu}}. \tag{6.68}$$

This is equivalent to the standard formulation of Equation 2.33 with $\beta = 0$ by a computation that $d\mathcal{S}_{g_{\mu\nu}} = G_{\mu\nu} = R_{\mu\nu} - 1/2sg_{\mu\nu}$ which can be found in [6].

Einstein initially found this theory by searching for a natural symmetric two-tensor built from the metric which was guaranteed to be divergence free. The fact that $R_{\mu\nu} - 1/2sg_{\mu\nu}$ satisfies $\delta_{g_{\mu\nu}} G_{\mu\nu} = 0$ can most naturally be seen from the above by observing that S is invariant under $Diff(X^n)$; hence $d\mathcal{S}_{g_{\mu\nu}}|_{Diff(X^n)} = 0$ and the conclusion follows from 6.62.

Now in this formulation, only the Levi-Civita connection on the frame bundle is considered. As such, there is nothing particularly interesting about dimension 4. If one allows for the possibility of torsion however, the situation changes.

Let us assume that we are given a metric connection $A \in \mathcal{A}(P_{Fr}(X^4_{g_{ij}}))$ with (possibly) non-zero torsion. In this case we could (for example) posit that $T_{\mu\nu}$ be a linear combination of $\Pi_{\Lambda^4}(F_A)$ and $\Pi_{Y_3}(F_A)$ such as $T_{\mu\nu} = *\Pi_{Y_3 \oplus \Lambda^4}(F_A) - \frac{1}{2} * \Pi_{\Lambda^4}(F_A)$ and ask for solutions of equations like

$$G_{\mu\nu} = *\Pi_{Y_3 \oplus \Lambda^4}(F_A) - \frac{1}{2} * \Pi_{\Lambda^4}(F_A). \tag{6.69}$$

Similarly, as we have two scalar summands of F_A, we could consider a Lagrangian of the form

$$L(\tau_A) = \int_{X^4} (s(g_{\mu\nu})\ dvol_{g_{\mu\nu}} + \Pi_{\Lambda^4}(F_A)). \tag{6.70}$$

The author suspects that a satisfactory elliptic theory (for Euclidian signature) might have to require an additional vanishing condition such as $\Pi_{\Lambda^2}(F_A) = 0$ together with a vanishing condition in the torsion like $\Pi_{Y_2^\pm}(\tau_A)$. But this is purely speculative.

6.6.2 Other Peculiarities of Torsion and Curvature summands in Low Dimensions for P_{Fr}

Dimension 5

The irreducible summands of the torsion of a Riemannian 4-fold $\Lambda^1, Y_2, \Lambda^3$ are dual to the summands $\Lambda^4, Y_3, \Lambda^2$. This Allows one to investigate the Vector field $*\Pi_{\Lambda^4 \oplus Y_3 \oplus \Lambda^2}(F_A)$ on $\mathcal{A}(P_{Fr})$.

Dimension 6

Dimension 6 is distinguished in the following ways. The Λ^3 summand of the torsion is a complex representation as is the Y_3 representation in the curvature. The Λ^2 and Λ^4 representation are hodge dual to each other as well.

Dimension 7

The Λ^3 summand of the torsion is dual to the Λ^4 summand of the curvature.

Dimension 8

Dimension 8 is distinguished in that metric connections on P_{Fr} carry an intrinsic notion of chiral curvature. To see this we recall that the algebraic Bianchi identity

$$R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kij}{}^{l} = 0 \tag{6.71}$$

for the curvature tensor of the Levi-Civita connection is equivalent to the assertion that $\Pi_{\Lambda^4}(R_{ijk}{}^{l}) = 0$. This is not valid if the connection is metric with non-zero torsion. In this case $\Pi_{\Lambda^4}(R_{ijk}{}^{l})$ is irreducible unless the dimension is 8. Here this summand decomposes as

$$\Pi_{\Lambda^4}(R_{ijk}{}^{l}) = \Pi_{\Lambda^4_+}(R_{ijk}{}^{l}) \oplus \Pi_{\Lambda^4_-}(R_{ijk}{}^{l}) \tag{6.72}$$

where both summands on the right hand side are irreducible. This has the further implication that the summands of the complete rolled up half-signature complex live with in the Torsion and Curvature summands for P_{Fr}.

6.6.3 $SU(3)$-Structures in Dimension 6

We point out that the Dolbeaut sequence of representations in dimension 6 for a space with a(n) (almost) complex structure and a(n) (almost) complex orientation is the complex analog of the DeRahm sequence in real dimension 3. It appears that one could attempt to develop a Yang-Mills-Higgs monopole theory along the lines discussed in Atiyah and Hitchen's monograph on monopoles for this restricted class of spaces. The most notorious example of such a space is $S^6 \cong G_2/SU(3)$ for which it is unknown whether the space carries and integrable complex structure.

6.6.4 G_2-Structures in Dimension 7

Casson's invariant of a Homology 3 sphere Σ^3 was reinterpreted by Taubes to be the "Euler Carachteristic" of a vector field on the Space of Connections of the Trivial Principal $SU(2)$-bundle $P_{SU(2)} = \Sigma^3 \times SU(2)$.

If we assume that we are given a 7-manifold $(X^7_{G_2}, g_{ij})$ whose frame bundle P_{Fr} has been reduced via the canonical inclusion $\iota : G_2 \hookrightarrow SO(7)$ to a bundle P_{G_2}, then we can ask for the zeros of the "vector field" $\Pi_{\mathbb{R}^7}(F_A)$. While the analogous elliptic machinery exists (assuming the G_2 structure), the geometric problems seem far more complicated.

6.6.5 $Spin(7)$-Structures in Dimension 8

Consider the real 8-dimensional Spin-representation of $Spin(7)$

$$\Delta : Spin(7) \longrightarrow SO(8) \tag{6.73}$$

and consider an 8-dimensional Riemannian manifold $(X^8_{Spin(7)}, g_{ij})$ whose structure bundle of frames P_{Fr} has been reduced from $SO(8)$ to $Spin(7)$ via Δ. Now on such a manifold, the two forms decompose as the direct sum

$$\Lambda^2(\mathbb{R}^8) \cong \mathbb{R}^7 \oplus \Lambda^2(\mathbb{R}^7). \tag{6.74}$$

This makes it possible to define the equation

$$F_A = \Pi_{\Lambda^2(\mathbb{R}^7)}(F_A) \tag{6.75}$$

for a connection $A \in \mathcal{A}(P_G)$ on an auxiliary principal G-bundle P_G over $(X^8_{Spin(7)}, g_{ij})$. This equation is well defined on Gauge equivalence classes and thus cuts out a submanifold $\mathcal{M}(P_G) \subset \mathcal{A}(P_G)/\mathcal{G}(P_G)$.

Analogously to the 4-dimensional situation, the virtual tangent space to $[A] \in \mathcal{M}$ is given at $A \in \mathcal{A}$ as H^1 of the complex

$$0 \longrightarrow \Omega^0 \otimes Ad(P_G) \xrightarrow{d_A} \Omega^1 \otimes Ad(P_G) \xrightarrow{\Pi_{\mathbf{7}} \circ d_A} \Omega^2_{\mathbf{7}} \otimes Ad(P_G) \longrightarrow 0. \quad (6.76)$$

This complex is elliptic with symbol given by Octonionic multiplication; non-degeneracy is assured by the fact that $\mathbb{O}$ is a division algebra. The complex 6.76 when rolled up is likewise equivalent to the twisted Dirac operator

$$v\partial\!\!\!/_A \colon S^+ \otimes Ad(P_G) \longrightarrow S^- \otimes Ad(P_G). \quad (6.77)$$

As such, the index of 6.76 gives the dimension of $\mathcal{M}$ as

$$Dim(\mathcal{M}) = << \hat{A}(X^8_{Spin(7)}) \cdot ch(Ad(P_G)), [X^8_{Spin(7)}] >> \quad (6.78)$$

assuming that $H^0 = H^2 = 0$. This theory is limited by two main factors. Firstly, that ellipticity rests on the notion of a division algebra. As these algebras occur only in dimensions 1,2,4 and 8, it can be easily seen that there are no other theories of this form as dimensional considerations prohibit. Secondly, the class of 8-manifolds admitting a $Spin(7)$ structure are limited by considerations of characteristic classes. In order for a manifold to carry such a structure, its tangent bundle must satisfy a somewhat restrictive equation in its characteristic classes. While there are many important 8-manifolds which satisfy this equation (e.g. $\mathbb{H}P^2$), the sphere S^8 is not one of them. Thus we do not have a model for the fundamental solutions of this theory.

We point out that by an exceptional isomorphism, $SU(4) \cong Spin(6)$. If one considers a Calabi-Yau manifold to be a Kahler manifold whose structure bundle and holonomy have been reduced to $SU(n)$ then it is easy to see that in real Dimension 8, a Calabi-Yau space is necessarily a $Spin(7)$ manifold. For this class of $Spin(7)$ spaces, the above theory is actually a generalization of the theory of Degree 0 stable bundles in the sense that there is an inclusion from the moduli space of Degree 0 stable bundles into the space of Connections with $\Pi_{\Lambda^2(\mathbb{R}^7)}(F_A)$. This is demonstrated by using the Work of Uhlenbeck and Yau showing that on any stable bundle, there is a choice of connection with constant curvature along the Kahler form. In the case of Degree 0 bundles, the curvature vanishes identically along the Kahler form forcing the curvature to lie in the preferred summand relative to the $Spin(7)$ structure.

6.7 Other Elliptic Symbol Sequences

The reader has probably suspected by now that the decompositions 6.24 and 6.59 are in fact related to some more general phenomena. We should probably point out that this work is some sense a first application of a more ambitious project to develop a constellation of co-homology theories generalizing the DeRahm theory using a notion of generalized ellipticity of the type considered by Douglis, Nirenberg and Agmon. We excerpt below from a letter describing the general outline of the approach (sent to Raoul Bott).

Dear Raoul,

The object of this note is to describe a sequence of vector bundles over a Riemannian manifold $(X^n, g_{i,j})$ which I think you might find interesting. I think that these sequences may be related to cohomolgy theories via a generalization of the notion of an elliptic complex, where not all operators in the complex are of the same order. This is a substantial generalization of the "Y_i" sequence ... which we discussed on the last day of my trip [Jan-Feb 1992].

For simplicity, let me assume that we are on a $Spin(2m)$ manifold for the moment so that it will be sufficient to specify these bundles by representations of the lie algebra of $Spin(2m)$. The sequences of bundles are functorially associated to Quadruples, $(X^{2m}, P_{Spin(2m)}, V, x_0)$ consisting of a manifold X, a spin structure given by a principal $Spin(2m)$ bundle $P_{Spin(2m)}$ covering the bundle of orthonormal frames, an irreducible representation V of $Spin(2m)$, and a non negative integer x_0. All representations should be taken complex when necessary.

Let us say we are given an irreducible representation V_λ of highest weight λ of $Spin(2m)$ together with a non-negative integer x_0 which we will refer to as the level. Recall that the highest weights of the irreducible representations are in one to one correspondence with the with maps from the set of simple *roots into the non-negative integers. We may find it convenient to represent a given irreducible Spin(2m)-module by such an integral labeling of the nodes of the appropriate Dynkin diagram. As Spin(2m) corresponds to the D_m diagram, let us fix the notation that an ordered m-tuple of non-negative integers $(x_1, x_2, .., x_i, .., x_{m-1}, x_m)$ corresponds to the labeling where x_i is assigned to the i^{th} node (going from left to right) as long as $i \leq m-2$ and x_{m-1} is assigned to the upper node (or positive-fork) and x_m is assigned to the lower node (negative fork).*

We will now specify irreducible representations of Spin(2m), $\{\rho_i^{\lambda,x_0}\}_{i=0}^{m-1}$ and a reducible representation $\rho_m^{\lambda,x_0} \cong \rho_{m,+}^{\lambda,x_0} \oplus \rho_{m,-}^{\lambda,x_0}$ whose summands are irreducible. In this sequence $\rho_0^{\lambda,x_0} = V_\lambda$. Explicitly:

$$\begin{aligned}
\rho_0^{\lambda,x_0} &= (x_1, x_2, .., x_i, .., x_{m-1}, x_m) \\
\rho_1^{\lambda,x_0} &= (x_0 + 1 + x_1, x_2, .., x_i, .., x_{m-1}, x_m) \\
\rho_i^{\lambda,x_0} &= (x_0, x_1, ..., x_{i-2}, x_{i-1} + 1 + x_i, x_{i+1} ..., x_{m-1}, x_m) \\
\rho_{m-1}^{\lambda,x_0} &= (x_0, x_1, .., x_i, .., x_{m-3}, x_{m-2} + 1 + x_{m-1}, x_{m-2} + 1 + x_m) \\
\rho_{m,+}^{\lambda,x_0} &= (x_0, x_1, .., x_i, .., x_{m-3}, x_{m-2} + 2 + x_{m-1} + x_m, x_{m-2}) \\
\rho_{m,-}^{\lambda,x_0} &= (x_0, x_1, .., x_i, .., x_{m-3}, x_{m-2}, x_{m-2} + 2 + x_{m-1} + x_m)
\end{aligned}$$

And trivialy we define $\rho_i^{\lambda,x_0} \cong \rho_{2m-i}^{\lambda,x_0}$ for $i \geq m$.

While this way of expressing vector bundles might appear slightly unfamiliar at first, it should be noted that not all of these sequences are unfamiliar as the deRahm cohomology corresponds to the sequence $\Omega^i(T^*X^{2m}) \cong \rho_i^{\lambda,x_0}$ gotten from the trivial representation $\lambda = 0$, at level $x_0 = 0$. Further, the Y_i sequence is in correspondence with the ρ_i^{λ,x_0}-series determined by $\lambda = 0$, at level $x_0 = 1$.

So first of all I wanted to know if this is some well known series. The reason I ask is that there are several interesting things about this sequence.

The first striking fact is that the Weyl dimension formula gives

$$\sum_{i=0}^{2m} (-1)^i \; Dim(\rho_i^{\lambda,x_0}) = 0.$$

Secondly, just as in the case of the DeRahm complex, this sequence further breaks into two 'half signature' complexes. These have the stronger property that separately:

$$\sum_{i=0}^{m,+}(-1)^i \; Dim(\rho_i^{\lambda,x_0}) = 0$$

$$\sum_{i=0}^{m,-}(-1)^i \; Dim(\rho_i^{\lambda,x_0}) = 0.$$

Further, the computer says that the 'alternating sum' of these bundles in the sense of K-theory, when pulled up to the unit sphere bundle of $T^*(X)$, is trivial. Lastly, it appears that we always have

$$\rho_{i+1}^{\lambda,x_0} \subset S_0^{x_i+1}(T^*(X^{2m})) \otimes \rho_i^{\lambda,x_0}$$

and that it is this heterogeneous sequence of 'symbol' maps which realizes the isomorphism. Here, $S_0^{x_i+1}(T^*(X^{2m})) \subset S^{x_i+1}(T^*(X^{2m}))$ is the sub-bundle of the symmetric bundle defined by the irreducible representation of $Spin(2m)$ whose highest weight is $x_i + 1$-times the highest weight of the defining representation

$$\mu : Spin(2m) \longrightarrow SO(2m).$$

Thus we appear to be defining, via a generalized exact symbol sequence, an element of the compactly supported K-theory of $T^*(X)$.

The big difference with the standard notion of elliptic complexes is that the order of the operators used is not uniform. For instance, our Y_i sequence starts off with a second order operator and is then followed by first order operators. This might not be so crazy since ... the sum of the principal symbols of a heterogeneous complex is not the same as the principal symbol of the corresponding rolled up operator; these coincide only with if one makes the usual homogeneity assumption.

However there are some interesting instances where these ρ_i sequences have homogeneous order. For example, consider the usual dirac operator comming from the Levi-Civita connection A:

$$\partial\!\!\!/_A : S^+ \longrightarrow S^-.$$

Via representation theory, this operator is defined by noticing that

$$S^+ \otimes T^* \cong S^- \oplus Tw^-$$

where $Tw^- = C(S^+, T)$ is the irreducible "twistor" representation which is most easily described as the above cartan product. We can express the Dirac operator as the composition:

$$\partial\!\!\!/_A : S^+ \xrightarrow{\nabla_A} S^+ \otimes T^* \xrightarrow{\Pi_{S^-}} S^-.$$

Now instead of following covariant differentiation by projection onto the subspace of negative spinors, we could project onto the orthogonal compliment Tw^- which yields an operator

$$\hat{\partial\!\!\!/}_A : S^+ \longrightarrow Tw^-$$

which is often refered to as the "Twistor operator". Unlike the Dirac operator however, this operator is not elliptic. However we can extend it to an elliptic complex (at least at the level of its symbol sequence) by viewing it as the first operator in the sequence corresponding to $V_\lambda = S^+$ and level $x_0 = 0$.

Conversely, the co-twistor operator

$$\hat{\partial\!\!\!/}^*_A : S^- \longrightarrow Tw^+$$

gotten from the adjoint of the Dirac operator

$$\partial\!\!\!/^*_A : S^- \longrightarrow S^+$$

can likewise be extended. All operators here are first order. It had never occured to me that untwisted spinors might appear in other elliptic complexes/operators. Here one has two candidate elliptic complexes extending the twistor operators with (elliptic) maps between the candidate chain groups where the first such map is the usual Dirac operator. Two ways in which this might be useful are to try and explain:

I. Why the Dirac operator is rigid while its kernel and co-kernal are not.

II. Why the Dirac operator does not give a topologically invariant two step cohomology theory.

by including the Dirac operator inside of larger elliptic operators and complexes.

One other thing occurs to me. I was thinking about your opening remarks in your paper with Cliff on the Witten rigidity conjectures. You remark that the G-index of the two step DeRahm and Signature operators is rigid as *they are given as rolled up elliptic complexes whose cohomology is homotopy invariant as the harmonic forms stay put. Then you go on to prove that two other sequences of operators are universally rigid corresponding to formal generalizations of the Signature and Dirac operators (requiring $w_2 = 0$ and $p_1 = 0$ respectively). This is preceded by the remark that "even the first new candidate", corresponding to tensoring the signature operator by the tangent bundle, cannot be proven to be rigid in isolation.*

So I was thinking, that you don't unroll these operators but treat them as two step complexes. However if you unroll this first operator, and subtract 2 unrolled signature operators from it (which are already known to be rigid) then you get back exactly our Y_i sequence and are proving a theorem which might be made more concrete if the Y_i sequence were giving rise to a homotopy insensitive cohomology theory.

So I started trying some of your higher operators above the first level, and it seems to me that for any of your operators I can give you a formal sequence of these rolled up ρ_i^{λ,x_0} bundles with operators which is equivalent to your twisted signature operators.

The point is that maybe you are proving theorems about combinations of cohomology theories and that these combinations in particular are rigid.

Various ones of the above sequences exist for O(n), SO(n) and Pin(n) structures a well as analogues for U(n) and SU(n) and Sp(n)Sp(1). The question is whether these symbol sequences should come from an honest differential with $d^2 = 0$ gotten from successive covariant differentiation using the Levi-Civita connection followed by projection. The dimensional numerology seems compelling to me but I have not really considered these much beyond what I am telling you...I am wondering what you make of all this.

All The Best,
Eric

[Sent July 18 to Svetlana@math.harvard.edu. Sent in previous version: May 3rd 1992 to gordana@joe.math.uga.edu]

From this point of view, the "complexes" 6.24 and 6.59 come from the decompositions of the above "inch-worm" sequences for $Sl(8,\mathbb{R})$ (corresponding to the A_7 Dynkin diagram) under the canonical $SO(8)$ subgroup. The author requests permission to work the theory out somewhat further before saying anything more.

6.8 Spin(10)-Phenomenology

We would like to finish by discussing briefly the physical motivation for the present work.

Let $(X^{1,3}, g_{ij})$ be a Lorentzian space time with spin structure and P_G be a Principal G-bundle over $X^{1,3}$ with $E = P_G \times_\rho V$ an associated vector bundle.

In what is currently regarded as the 'Standard model' the particles of nature which make up ordinary matter at its most basic level are modeled by sections of a bundle of the form $(S^+ \oplus S^-) \otimes E$ and are refered to as (fundamental) fermions. The standard model takes for its group G

$$G = SU(3) \times SU(2) \times U(1) \tag{6.79}$$

and E to contain a 16 dimensional (or at the least 15 dimensional, depending on whether neutrinos are assumed massive) complex representation of G. In the full theory, the 16 dimension dimensional (sub)-representation is repeated three times corresponding to three families of fundamental fermions.

Now let us assume that we are given a Riemannian immersion

$$\iota : (X^{1,3}, g_{ij}) \longrightarrow (X^{1,3+j}, g_{ij}) \tag{6.80}$$

where we assume that $j \geq 10$. If we took the Dirac operator on the total space and restricted it to $\iota(X^{1,3})$ then it would appear as a twisted Dirac operator

$$\partial\!\!\!/_A \colon S^+(X^{1,3}) \otimes E \longrightarrow S^-(X^{1,3}) \otimes E \tag{6.81}$$

where the coefficient bundle

$$E = (S^+(N_\iota) \oplus S^-(N_\iota)) \tag{6.82}$$

would be the total spin bundle of the normal bundle N_ι and there would be a (presumably slight) correction factor from the second fundamental form of the immersion.

For simplicity, we will discuss the case j=10. Let us make the assumption that we can reduce the structure group of N_ι via the chain

$$G \subset SU(5) \subset Spin(10) \tag{6.83}$$

where the inclusions are those familiar from the Georgi-Glashow Grand unified theory. In this case we may take the structure group of the normal bundle to be P_G and E to be as in 6.82. It is striking to notice that we recover essentially the correct quantum numbers (representations) from the splitting of the ordinary Dirac operator under the restriction to the immersed sub-manifold.

In summary, the "SO(10)"-particle phenomenology, is perceived by us as a strong *geometric* motivation for the existence of sophisticated equations governing Spin-bundles in higher dimensions. The ability to find 1^{st} order Yang-Mills equations for these bundles capturing something of the flavor of the 4-dimensional theory seems encouraging to this end.

Appendix A

Determinants

A.1 Mathematica Code Generating the Symbol Matrix [1]

```
dim=8; (* The dimension we are at *)
v=w[1]; (* The cotangent vector *)

Clear[linear] (* linear[op] declares op to be linear,
similarly for mlinear *)
linear[op_]:=(op[0]=0;
      op[x_Symbol * y_]:=x*op[y];
      op[x_?NumberQ * y_]:=x*op[y];
      op[x_+y_]:=op[x]+op[y])
Clear[mlinear]
mlinear[op_]:=(op[lft___,0,rgt___]=0;
op[lft___,x_?NumberQ * y_,rgt___]:=x*op[lft,y,rgt];
op[lft___,x_Symbol * y_,rgt___]:=x*op[lft,y,rgt];
op[lft___,x_+y_,rgt___]:=op[lft,x,rgt]+op[lft,y,rgt])

Clear[n] (* The "mod out by relations" operator *)
SetAttributes[n,Listable]
```

[1] Available from eric@humus.huji.ac.il

```
linear[n]
n[T[x___]]:=n /@ T[x]

(* w is the wedge container *)
n[w[x___]]:=Signature[List[x]]*(w @@ Union[List[x]])

Clear[T] (* The tensor product container *)
mlinear[T]

Clear[flip] (* commutativity of tensor products *)
linear[flip]
flip[x_~T~y_]:=y~T~x

Clear[e] (* The exterior product operator *)
linear[e]
e[T[x___w]]:=Flatten[w[x],1] // n

Clear[star] (* The star operator *)
linear[star]
star[w[x___]]:=
Module[{ans},
  ans=Complement[Range[dim],{x}];
  Signature[{x}~Join~ans]*(w @@ ans)
]

Clear[i] (* The interior product operation *)
linear[i]
i[T[x_,y_]]:=star[e[T[x,star[y]]]]
(* Splitting a dim/2-form into a positive and a negative
   part *)
Clear[cleave,cleavep,cleavem]
linear[cleave]; linear[cleavep]; linear[cleavem]
cleave[(x_)?(Head[#1] == w && Length[#1] == dim/2 & )] :=
{x + star[x], x - star[x]}/2
cleavem[x_?((Head[#]==w && Length[#]==dim/2)&)]:=
(x-star[x]) / 2
cleavep[x_?((Head[#]==w && Length[#]==dim/2)&)]:=
```

```
(x+star[x]) / 2

Clear[raise] (* the map ^n x ^m -> ^(n+1) x ^(m+1) *)
linear[raise]
raise[T[x_w,y_w]]:=Sum[e[x~T~w[i]]~T~e[w[i]~T~y],{i,dim}]

Clear[shift] (* the map ^n x ^m -> ^(n+1) x ^(m-1) *)
linear[shift]
shift[T[x_w,y_w]]:=raise[T[x,star[y]]] /. {lft_~T~rgt_
-> lft~T~star[rgt]}

Clear[lower] (* the map ^n x ^m -> ^(n-1) x ^(m-1) *)
linear[lower]
lower[T[x_w,y_w]]:=raise[star[x]~T~star[y]] /.
{lft_~T~rgt_ -> star[lft]~T~star[rgt]}

Clear[projY] (* The map ^1 x ^n -> Yn *)

linear[projY]
projY[w[x_]~T~y_w]:=Module[{ans,n},
  n=Length[y];
  ans=w[x]~T~y-raise[w[]~T~i[w[x]~T~y]]/(dim+1-n);
  ans=ans-(-1)^(n dim)*shift[w[]~T~e[w[x]~T~y]]/(n+1);
  Expand[ans]
]

(* The maps *)
Clear[dee,delta] (* differentiation maps *)
linear[dee]
dee[x_w~T~y_]:=e[v~T~x]~T~y
linear[delta]
delta[x_w~T~y_]:=i[v~T~x]~T~y

(* Indeterminates *)
Clear[c1L3,c2L3,c1L4p,c2L4p,c3L4p,c1Y3,c2Y3,c1L2,c2L2,c3L2]

Clear[iY4p,iL3,iY2,iL1]  (* Injections *)
```

```
iY4p[x_]:=x
linear[iL3]
iL3[x_?((Head[#]==w && Length[#]==3)&)]:=Expand[
c1L3*(raise[w[]~T~x] /. {lft_~T~rgt_ -> lft~T~cleavep[rgt]})
+c2L3*shift[w[]~T~x]]
iY2[x_]:=x
linear[iL1]
iL1[x_?((Head[#]==w && Length[#]==1)&)]:=Expand[raise[w[]~T~x]]

Clear[pL4p,pY3,pL2,pY1] (* Projections *)
linear[pL4p]
pL4p[T[x_?((Head[#]==w && Length[#]==2)&),
y_?((Head[#]==w && Length[#]==4)&)]]:=
c1L4p*i[flip[shift[y~T~x]]]
pL4p[T[x_?((Head[#]==w && Length[#]==2)&),
y_?((Head[#]==w && Length[#]==2)&)]]:=
c2L4p*cleavep[e[x~T~y]]
pL4p[T[x_?((Head[#]==w && Length[#]==0)&),
y_?((Head[#]==w && Length[#]==4)&)]]:=
c3L4p*x~T~y /. {w[]~T~z_ -> z}
pL4p[T[x_?((Head[#]==w && Length[#]==0)&),
               y_?((Head[#]==w && Length[#]==2)&)]]:=0

linear[pY3]
pY3[T[x_?((Head[#]==w && Length[#]==2)&),
               y_?((Head[#]==w && Length[#]==4)&)]]:=
Expand[c1Y3*projY[lower[x~T~y]]]
pY3[T[x_?((Head[#]==w && Length[#]==2)&),
               y_?((Head[#]==w && Length[#]==2)&)]]:=
Expand[c2Y3*projY[flip[shift[x~T~y]]]]
pY3[T[x_?((Head[#]==w && Length[#]==0)&),
               y_?((Head[#]==w && Length[#]==4)&)]]:=0
pY3[T[x_?((Head[#]==w && Length[#]==0)&),
               y_?((Head[#]==w && Length[#]==2)&)]]:=0

linear[pL2]
pL2[T[x_?((Head[#]==w && Length[#]==2)&),
```

```
                 y_?((Head[#]==w && Length[#]==4)&)]]:=
         Expand[c1L2*i[x~T~y]]
pL2[T[x_?((Head[#]==w && Length[#]==2)&),
                 y_?((Head[#]==w && Length[#]==2)&)]]:=
         Expand[c2L2*e[lower[x~T~y]]]
pL2[T[x_?((Head[#]==w && Length[#]==0)&),
                 y_?((Head[#]==w && Length[#]==4)&)]]:=0
pL2[T[x_?((Head[#]==w && Length[#]==0)&),
                 y_?((Head[#]==w && Length[#]==2)&)]]:=c3L2*y

linear[pY1]
pY1[T[x_?((Head[#]==w && Length[#]==2)&),
                 y_?((Head[#]==w && Length[#]==4)&)]]:=0
pY1[T[x_?((Head[#]==w && Length[#]==2)&),
                 y_?((Head[#]==w && Length[#]==2)&)]]:=
         Expand[projY[lower[x~T~y]]]
pY1[T[x_?((Head[#]==w && Length[#]==0)&),
                 y_?((Head[#]==w && Length[#]==4)&)]]:=0
pY1[T[x_?((Head[#]==w && Length[#]==0)&),
                 y_?((Head[#]==w && Length[#]==2)&)]]:=0

Clear[sp] (* The scalar product *)
SetAttributes[sp,Listable]
mlinear[sp]
Print["Warning! sp behaves well only on normalized operands!!"]
sp[x_T,y_T]:=If[x===y,1,0]
sp[x_w,y_w]:=If[x===y,1,0]
sp[x_w,y_T]:=0
sp[x_T,y_w]:=0

(* Bases *)
Clear[kertr,baseL1,baseY1,baseL2,baseY2,baseL3,baseY3,
baseL4p,baseY4p]
kertr[d_]:=kertr[d]=Table[RotateRight[{1,-1}~Join~Table[0,
{d-2}],i],{i,0,d-2}]
baseL1=w /@ Range[dim]
baseY1=Expand[n[
```

```
Flatten[Table[w[i]~T~w[j]+w[j]~T~w[i],{i,dim-1},{j,i+1,
dim}]]~Join~
Table[w[i]~T~w[i]-w[i+1]~T~w[i+1],{i,7}]]]
baseL2=Expand[n[Flatten[Table[w[i,j],{i,dim-1},{j,i+1,dim}]]]]
baseY2=Expand[n[
Flatten[Table[kertr[3].(List @@ shift[w[]~T~w[i,j,k]]),
{i,dim-2},{j,i+1,dim-1},{k,j+1,dim}]]~Join~
Flatten[Table[kertr[7].(List @@ raise[w[]~T~w[i]]),{i,8}]]]]
baseL3=Expand[n[Flatten[Table[w[i,j,k],{i,dim-2},{j,i+1,dim-1},
{k,j+1,dim}]]]]
baseY3=Expand[n[
Flatten[Table[kertr[4].(List @@ shift[w[]~T~w[i,j,k,l]]),
                {i,dim-3},{j,i+1,dim-2},{k,j+1,dim-1},
{l,k+1,dim}]]~Join~
Flatten[Table[kertr[6].(List @@ raise[w[]~T~w[i,j]]),
{i,dim-1},{j,i+1,dim}]]]]
baseL4p=Expand[n[Flatten[
Table[cleavep[w[i,j,k,l]],{i,4},{j,i+1,5},{k,j+1,6},
{l,k+1,7}]]]]
baseY4p=Expand[n[
Flatten[Table[Module[{t1,t2},
  t1=List @@ raise[w[]~T~w[i,j,k]];
  t2=(t1 /. {lft_~T~rgt_ -> lft~T~star[rgt]});
  kertr[5].(t1+t2)
],{i,dim-2},{j,i+1,dim-1},{k,j+1,dim}]]]]

(* finally, the symbol map *)
Clear[matline,mat,m]
matline[x_]:=Module[{bp},
bp=dee[x]+delta[x];
Join[sp[Expand[n[pL4p[bp]]],baseL4p],
sp[Expand[n[pY3[bp]]],baseY3],
sp[Expand[n[pL2[bp]]],baseL2],
sp[Expand[n[pY1[bp]]],baseY1]]]
mat[base_]:=Table[Module[{p},Print[i1];
Print[p=matline[base[[i1]]]];p],{i1,Length[base]}]
m=Join[mat[iL1 /@ baseL1],mat[iL3 /@ baseL3],
```

```
mat[iY2 /@ baseY2],
mat[iY4p /@ baseY4p]];
Save["mat.new",m]
(* dimensions:
baseL1:   8 baseLp4:  35
baseL3:  56 baseY3: 350
baseY2: 160 baseL2:  28
baseY4p: 224 baseY1:  35 *)
```

A.2 Example of an Input File Program

```
<< mat
c1L3=1;c2L3=0;c1L4p=1;c2L4 =1;c3L4p=-1;c1Y3=1;c2Y3=1;
c1L2=1;c2L2=1;c3L2=-1
Print[{c1L3,c2L3,c1L4p,c2L4p,c3L4p,c1Y3,c2Y3,c1L2,c2L2,c3L2}]
m=2*m
Print[Det[m]]
Quit
```

A.3 Determinants For Various Sets of Constants

Below we list the determinants of six different sets of constants. The first five of which are non-zero with the 6^{th} included as an example of a non-elliptic choice of equations. The 5^{th} example is one in which the injection of the frame torsion into the admissible torsion is generated naturally by the inclusion of Lie algebras $so(8) \hookrightarrow su(8)$ or equivalently $\Lambda^2 \hookrightarrow \Lambda^2 \oplus \Lambda^4_{\pm}$. Note:Determinants have been factored using the FactorInteger command.

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- Terminal graphics initialized --
{1, 1, 1, 1, 0, 1, 1, 1, 1, 1}
```

$$Det = -1 \cdot 2^{491} \cdot 3^{43} \cdot 5^{21} \cdot 7^{29} \tag{A.1}$$

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- Terminal graphics initialized --
{-2, -2, 2, -2, 3, 3, -3, -1, 0, 1}
```

$$Det = -1 \cdot 2^{547} \cdot 3^{414} \cdot 5^{42} \cdot 7^{8} \cdot 29^{35} \tag{A.2}$$

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- Terminal graphics initialized --
{1, 1, 1, 1, -1, 1, 1, 1, 1, -1}
```

$$Det = -1 \cdot 2^{526} \cdot 3^{43} \cdot 5^{21} \cdot 7^{8} \cdot 11^{21} \tag{A.3}$$

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- Terminal graphics initialized --
{1, 0, 1, 1, -1, 1, 1, 1, 1, -1}
```

$$Det = -1 \cdot 2^{491} \cdot 3^{43} \cdot 5^{77} \cdot 7^{8} \tag{A.4}$$

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- Terminal graphics initialized --
{0, 1, 1, 1, -1, 1, 1, 1, 1, -1}
```

$$Det = -1 \cdot 2^{512} \cdot 3^{99} \cdot 5^{21} \cdot 7^{8} \tag{A.5}$$

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- Terminal graphics initialized --
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

$$Det = 0 \tag{A.6}$$

Bibliography

[1] Atiyah, M.F. *Geometry of Yang-Mills Fields*, Fermi Lectures Notes, Scuola Normale Sup. di Pisa, Pisa, 1979.

[2] Atiyah, M.F., Bott, R., and Shapiro, A. Clifford Modules, *Topology 3, Suppl. 1,* (1964), 3-38.

[3] Atiyah, M.F., Hitchin, N., and Singer, I.M. Self-Duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A. 362* (1978), 425-461.

[4] Atiyah M.F., and Singer, I.M. The Index of Elliptic Operators III, *Ann. of Math. 87* (1968), 546-604.

[5] Berline, N., Getzler, E., and Vergne, M. *Heat Kernels and Dirac Operators*, Grundlehren der mathematischen Wissenschaften 298, Springer-Verlag, Berlin, 1992.

[6] Besse, A. L. *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete ; 3. Folge, Band 10, Springer-Verlag, Berlin, 1987.

[7] Bryant, R. Metrics With Exceptional Holonomy, *Ann. of Math. 126* (1987), 525-576.

[8] Donaldson, S.K., and Kronheimer, P.B., *The Geometry of Four Manifolds*, Oxford Mathematical Monographs, Clarendon Press, Oxford, (1990).

[9] Dubrovin, B.A., Fomenko, A.T., and Novikov, S.P., *Modern Geometry-Methods and Applications. Part III: Introduction to Homology Theory.*, Graduate Texts in Mathematics, Vol. 124, Springer-Verlag, Berlin, 1990.

[10] Glashow, S., and Weinstein, E. *Wager on the Containment (or Non-containment) of $SU(3)$ Within G_2*, April 1991.

[11] Gray, A., and Green, P. Sphere Transitive Structures And The Triality Automorphism, *Pacific Journal of Mathematics, Vol. 34, No. 1*, 1970, 83-96.

[12] Harvey, F.R. *Spinors and Calibrations*, Perspectives in Mathematics, Vol. 9, Academic Press, San Diego, 1990.

[13] Knapp, A.W. *Representation Theory of Semisimple Groups*, Princeton Mathematical Series 36, Princeton University Press, Princeton, 1986.

[14] Lawson, H.B., Jr. *The Theory of Gauge Fields in Four Dimensions*, American Mathematical Society, Providence, 1985.

[15] Lawson, H.B., Jr., and Michelsohn, M-L. *Spin Geometry*, Princeton Mathematical Series 38, Princeton University Press, Princeton, 1989.

[16] Poor W.A. *Differential Geometric Structures*, Mc Graw Hill, New York, 1981.

[17] Postnikov, M.M. *Lectures in Geometry: Semester V. Lie Groups And Lie Algebras.* Mir Publishers, Moscow, 1986.

[18] Rapoport, D., and Sternberg S. On the Interaction of Spin and Torsion, *Annals of Physics, Vol 158, No. 2*, 1984, 447-475.

[19] Weinstein, E. *Personal Communication*, Letter to Raoul Bott, Summer 1992.